Systems of Ordinary Differential Equations

Václav Mácha

University of Chemistry and Technology

Example Let Q be aggregate supply, p be the price level, and π be the expected rate of inflation. Further, let Q^* be the long-run sustainable level output. We deduce

$$p'(t) = h(Q^* - Q) + \pi(t) \text{ for some } h > 0.$$

Next, we assume adaptive expectations, i.e.

$$\pi'(t) = k(p'(t) - \pi(t)) \text{ for some } k > 0.$$

which turns into

$$\pi'(t) = kh(Q^* - Q).$$

Example Let Q be aggregate supply, p be the price level, and π be the expected rate of inflation. Further, let Q^* be the long-run sustainable level output. We deduce

$$p'(t) = h(Q^* - Q) + \pi(t) \text{ for some } h > 0.$$

Next, we assume adaptive expectations, i.e.

$$\pi'(t) = k(p'(t) - \pi(t)) \text{ for some } k > 0.$$

which turns into

$$\pi'(t) = kh(Q^* - Q).$$

Assume $Q = Q(t) = a + bp(t) - c\pi(t)$ where $a > 0$, $b > 0$ and $c > 0$.

Example Let Q be aggregate supply, p be the price level, and π be the expected rate of inflation. Further, let Q^* be the long-run sustainable level output. We deduce

$$p'(t) = h(Q^* - Q) + \pi(t) \text{ for some } h > 0.$$

Next, we assume adaptive expectations, i.e.

$$\pi'(t) = k(p'(t) - \pi(t)) \text{ for some } k > 0.$$

which turns into

$$\pi'(t) = kh(Q^* - Q).$$

Assume $Q = Q(t) = a + bp(t) - c\pi(t)$ where $a > 0$, $b > 0$ and $c > 0$. Consequently

$$\begin{aligned} p'(t) &= -hbp(t) + (1 + hc)\pi(t) - ha + hQ^* \\ \pi'(t) &= -khbp(t) + khc\pi(t) - kha + khQ^* \end{aligned}$$

Example Let Q be aggregate supply, p be the price level, and π be the expected rate of inflation. Further, let Q^* be the long-run sustainable level output. We deduce

$$p'(t) = h(Q^* - Q) + \pi(t) \text{ for some } h > 0.$$

Next, we assume adaptive expectations, i.e.

$$\pi'(t) = k(p'(t) - \pi(t)) \text{ for some } k > 0.$$

which turns into

$$\pi'(t) = kh(Q^* - Q).$$

Assume $Q = Q(t) = a + bp(t) - c\pi(t)$ where $a > 0$, $b > 0$ and $c > 0$. Consequently

$$\begin{aligned} p'(t) &= -hbp(t) + (1 + hc)\pi(t) - ha + hQ^* \\ \pi'(t) &= -khbp(t) + khc\pi(t) - kha + khQ^* \end{aligned}$$

Crucial question: for which choice of constants is this system stable?

In what follows, we tackle the system

$$x'(t) = Ax(t) + b(t) \tag{1}$$

where the unknown $x(t)$ is a vector

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

and A is n by n matrix and b is $n-$dimensional vector.

In what follows, we tackle the system

$$x'(t) = Ax(t) + b(t) \tag{1}$$

where the unknown $x(t)$ is a vector

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

and A is n by n matrix and b is $n-$dimensional vector.

Definition

The set of functions defined on $\mathbb{R}$ and solving (1) is called a *general solution*. One of this function is called a *particular solution*.

Higher order linear equation as a linear equation:

Example

- Rewrite

$$y'' + ky' + my = 0$$

as a system of first-order ODEs.

Higher order linear equation as a linear equation:

Example

- Rewrite

$$y'' + ky' + my = 0$$

  as a system of first-order ODEs.

- Rewrite the following in matrix notation:

$$\begin{aligned} x' &= x + y + z \\ y' &= 2z - x \\ z' &= 4y \end{aligned} \tag{2}$$

Higher order linear equation as a linear equation:

Example

- Rewrite

$$y'' + ky' + my = 0$$

as a system of first-order ODEs.

- Rewrite the following in matrix notation:

$$\begin{aligned} x' &= x + y + z \\ y' &= 2z - x \\ z' &= 4y \end{aligned} \tag{2}$$

- Express

$$\begin{aligned} x'' + 3x + 2y &= 0 \\ y'' - 2x &= 0 \end{aligned} \tag{3}$$

as a system of first order ODEs

Classification:

- $b \equiv 0$ – homogeneous equation (i.e., $x'(t) = A(x)$)
- b non-zero – non-homogeneous equation

Classification:

- $b \equiv 0$ – homogeneous equation (i.e., $x'(t) = A(x)$)
- b non-zero – non-homogeneous equation

Theorem

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n linearly independent solutions to the homogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t). \tag{4}$$

Then every solution to (4) can be expressed in the form

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + \ldots + c_n\mathbf{x}_n(t),$$

where $c_1, \ldots, c_n$ are real constants.

Definition

A set of solutions $\{\mathbf{x_1}, \ldots, \mathbf{x}_n\}$ that are linearly independent is called a *fundamental solution set* for (4).

As always, we assume that the solution is certain exponential with proper coefficients. In particular, consider

$$x' = Ax$$

and assume that the solution is in the form

$$x(t) = e^{\lambda t} v$$

where v is a vector with constant coefficients.

As always, we assume that the solution is certain exponential with proper coefficients. In particular, consider

$$x' = Ax$$

and assume that the solution is in the form

$$x(t) = e^{\lambda t} v$$

where v is a vector with constant coefficients.

Lemma

Let A be n by n matrix with n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ and with corresponding eigenvectors $v_1, \dots, v_n$. Then the fundamental solution set is

$$\left\{ e^{\lambda_1 t} v_1, \dots, e^{\lambda_n t} v_n \right\}.$$

Example

- Find the fundamental solution set of

$$x' = \begin{pmatrix} -\frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} x.$$

Real not-distinct eigenvalues (especially double roots):

Real not-distinct eigenvalues (especially double roots):

Examples

- Solve

$$\mathbf{x}' = A\mathbf{x}$$

where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Real not-distinct eigenvalues (especially double roots):

Examples

- Solve
$$\mathbf{x}' = A\mathbf{x}$$
where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- Solve
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x}.$$

Matrix exponential

Recall

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Similarly, let A be a square matrix. Then we write

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \ldots$$

Matrix exponential

Recall

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Similarly, let A be a square matrix. Then we write

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \ldots$$

Let $\mathbf{v}$ be an eigenvector and w be a generalized eigenvector. How about $e^{At}v$ and $e^{At}w$?

Complex eigenvalues:

Example

- Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \mathbf{x}.$$

Complex eigenvalues:

Example

- Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \mathbf{x}.$$

Theorem

If the real matrix A has complex eigenvalues $\alpha \pm \beta i$ with corresponding eigenvectors $\mathbf{a} + i\mathbf{b}$, then the two linearly independent real vector solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$e^{\alpha t} \cos \beta t \mathbf{a} - e^{\alpha t} \sin \beta t \mathbf{b}$$
$$e^{\alpha t} \sin \beta t \mathbf{a} + e^{\alpha t} \cos \beta t \mathbf{b}.$$

Non-homogeneous systems – method of undetermined coefficients

Exercise

- Solve

$$x' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} x + t \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$$

Non-homogeneous systems – method of undetermined coefficients

Exercise

- Solve
$$x' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} x + t \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$$

Lemma

Let the right hand side $b(t) = e^{rt} t^m f$ where $f \in \mathbb{R}^n$. Then one solution to

$$x' = Ax + b$$

is of the form

$$t^s e^{rt} \left(a_m t^m + a_{m-1} t^{m-1} + \ldots + t a_1 + a_0 \right)$$

where $s = 0$ if r is not a root of $\det(A - \lambda I)$, $s = 1$ if r is a single root of $\det(A - \lambda I)$, $s = 2$ if r is a double root of $\det(A - \lambda I)$ and so on.

Exercise

- Find all solutions to

$$x' = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{pmatrix} x + \begin{pmatrix} e^{-2t} \\ 2 \\ 1 \end{pmatrix}$$

Lemma

Let x_1 solves

$$x' = Ax + b_1$$

and let x_2 solves

$$x' = Ax + b_2.$$

Then $x_1 + x_2$ solves

$$x' = Ax + (b_1 + b_2).$$

Exercise A cosmetic manufacturer has a marketing policy based upon the price $x(t)$ of its salon shampoo. The production $P(t)$ and the sales $S(t)$ are given in terms of the price $x(t)$ and the change in price $x'(t)$ by the equations

$$P(t) = 4 - \frac{3}{4}x(t) - 8x'(t),$$
$$S(t) = 15 - 4x(t) - 2x'(t).$$

The differential equations for the price $x(t)$ and inventory level $I(t)$ are

$$x'(t) = k(I(t) - I_0), \qquad I'(t) = P(t) - S(t).$$

We can reformulate it as

$$x'(t) = k(I(t) - I_0),$$
$$I'(t) = \frac{13}{4}x(t) - 6kI(t) + 6kI_0 - 11.$$

Find the evolution of prices for $k = 1$, $I_0 = 50$ and initial values $x(0) = 10$ and $I(0) = 7$.

Systems in a plane: the unknown vector is in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

In general, the system can be written as

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{5}$$

Systems in a plane: the unknown vector is in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

In general, the system can be written as

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{5}$$

Exercises

- Solve

$$\begin{aligned} x' &= x \\ y' &= 2y \end{aligned} \tag{6}$$

-

$$\begin{aligned} x' &= -x \\ y' &= -2y \end{aligned} \tag{7}$$

Definition

If $x(t)$ and $y(t)$ is a solution pair to (5) for t in the interval I, then a plot in the $xy-$plane of the parametrized curve $(x(t), y(t))$ for $t \in I$, together with arrows indicating its direction with increasing t, is said to be a *trajectory* of the system. In such context we call the $xy-$plane the *phase plane*.

How to draw the trajectory? If y is a function of x, then

$$\frac{\partial y}{\partial x} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} = \frac{g(x, y)}{f(x, y)}$$

Definition

If $x(t)$ and $y(t)$ is a solution pair to (5) for t in the interval I, then a plot in the $xy-$plane of the parametrized curve $(x(t), y(t))$ for $t \in I$, together with arrows indicating its direction with increasing t, is said to be a *trajectory* of the system. In such context we call the $xy-$plane the *phase plane*.

How to draw the trajectory? If y is a function of x, then

$$\frac{\partial y}{\partial x} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} = \frac{g(x,y)}{f(x,y)}$$

Exercise

- Draw the trajectories for the two exercises from the previous slide.

Definition

A point $(x_0, y_0) \in \mathbb{R}^2$ where $f(x_0, y_0) = g(x_0, y_0) = 0$ is called a *critical point* (or *equilibrium point*) of the given system. The corresponding solution $x \equiv x_0$ and $y \equiv y_0$ is called an *equilibrium* solution (or *stationary solution*).

Lemma

Let $x(t)$ and $y(t)$ be a solution on $[0, \infty)$ to the given system where f and g are continuous. If the limits

$$\lim_{t\to\infty} x(t) = x_0, \quad \lim_{t\to\infty} y(t) = y_0$$

exist and are finite, then (x_0, y_0) is the critical point of the system.

Exercises

- Find the critical points and sketch the trajectories in the phase plane for

$$x' = -y(y-2)$$
$$y' = (x-2)(y-2)$$

  What is the behavior of the solution starting from $(3, 0)$, $(5, 0)$ and $(2, 3)$?

- Sketch several representative trajectories of

$$x' = \frac{3}{y}$$
$$y' = \frac{2}{x}$$

Reminder

Exercises

- Recall the example of Romeo and Juliet. Let R denotes Romeo's passion for Juliet ($R > 0$ means love, $R < 0$ is hate) and let J denote the love of Juliet toward Romeo. Let R and J be governed by

$$R' = J$$
$$J' = -R$$

  Sketch the trajectory starting at $(1, 1)$ into the phase plane.

- Find the critical points and the equation for trajectories for the system

$$x' = y - 1$$
$$y' = e^{x+y}.$$

- Find the critical points and draw by hand several representative trajectories for the system

$$x' = -8y$$
$$y' = 18x.$$

Sketch a phase diagram for

$$x' = 5x - 3y$$
$$y' = 4x - 3y.$$

Sketch a phase diagram for

$$x' = 5x - 3y$$
$$y' = 4x - 3y.$$

Types of equilibria:

Sketch a phase diagram for

$$x' = 5x - 3y$$
$$y' = 4x - 3y.$$

Types of equilibria:

stable node, unstable node, stable spiral, unstable spiral, saddle, center

Classification of linear systems

- Negative eigenvalues – stable node
- Positive eigenvalues – unstable node
- Negative and positive eigenvalue – saddle
- Purely imaginary eigenvalues – center
- Complex eigenvalues – spiral (stable if $Re\ \lambda < 0$, unstable if $Re\ \lambda > 0$).

Exercise

- Classify the equilibria of

$$x' = -5x + 2y + 5$$
$$y' = x - 4y - 1$$

  and sketh a phase diagram for this system.
- Classify the equilibria

$$x' = 5x - 3y + 9$$
$$y' = 4x - 3y - 6$$

Almost linear systems

Definition

An *almost linear system* is a system of the form

$$x' = a_{11}x + a_{12}y + f(x,y),$$
$$y' = a_{21}x + a_{22}y + g(x,y),$$

where f and g satisfies

$$\lim_{(x,y)\to 0} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0, \quad \lim_{(x,y)\to 0} \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0.$$

The system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is called the *corresponding linear system*.