Difference equations

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This chapter is devoted to the study of difference equations. Namely, we are looking for an unknown sequence $\{y(n)\}_{n=1}^{\infty}$ which fulfills

$$y(n+k) + p^1 y(n+k-1) + \ldots + p^k y(n) = a_n \tag{1}$$

where a_n is some given right hand side and $p^1, \ldots, p^k \in \mathbb{R}$ are given coefficients. Such equation is called 'linear difference equation of order k'.

Example: Assume that we have to pay a mortgage 200 000 USD. The interest of this mortgage is 0.1% per month and we pay monthly 1 000 USD. Let denote the sum we owe in the $n-$th month by $y(n)$. Clearly, $y(0) = 200\ 000$. What is the relation for $y(n)$? Can you compute it?

We proceed similarly as in the case of the linear differential equations with constant coefficients. First of all, we find all solutions to the homogeneous case

$$y(n+k) + p^1 y(n+k-1) + \ldots + p^k y(n) = 0$$

and then we find one particular solution to non-homogeneous equation. The sum of these two outcomes gives the set of all solutions to the given problem.

The assumed solution to the homogeneous problem is $y(n) = \lambda^n$. Thus, the characteristic equation is

$$\lambda^k + p^1\lambda^{k-1} + p^2\lambda^{k-2} + \ldots + p^k = 0.$$

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Theorem

Let $\{\lambda_j\}_{j=1}^k$ are real roots of the characteristic equation of multiplicity ν_k. Then the fundamental system is

$$\{n^\alpha \lambda_j^n,\ j \in \{1, \ldots k\},\ \alpha \in \{0, \ldots, \nu_j - 1\}\}.$$

Exercise:

- Find all solutions to

$$y(n+2)+4y(n+1)+3y(n)=0.$$

- Find all solutions to

$$y(n+2)-2y(n+1)+y(n)=0.$$

- Find the solution to

$$y(n+2)+y(n)=0$$

satisfying $y(0)=1$ and $y(1)=2$.

Special right hand side: Let $P(n)$ be a polynomial. One solution $y(n)$ of equation

$$L(y) = \alpha^n P(n)$$

is of the form

$$y(n) = n^m \alpha^n Q(n)$$

where $m = 0$ if α is not a root of the characteristic equation and m equals the multiplicity of the root α otherwise, and $Q(n)$ is a polynomial of degree at most $\deg P(n)$.

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Exercises

- Solve the mortgage exercise.
- Find the one solution to

  $$y(n+2) - 5y(n+1) + 6y(n) = n + 2 + 2^n$$

  satisfying $y(0) = -1$ and $y(1) = 1$.

Let go back to the mortgage example. We need to find one solution to

$$y(n+1) - 1.001y(n) = -1\ 000.$$

The right hand side is of the special form, namely $\alpha \equiv 1$ and $P(n) = 1\ 000$ is a polynomial of degree 0.

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$$y(n) = Q(n)$$

where $Q(n) = a \in \mathbb{R}$ since it can be only 0 degree polynomial.

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which yields $a = 1\ 000\ 000$. All solutions are of the form

$$y(n) = 1\ 000\ 000 + c1.001^n$$

and since $y(0) = 200\ 000$ we deduce

$$y(n) = 1\ 000\ 000 - 800\ 000 \cdot 1.001^n.$$

Complex roots once again

If $\lambda = a \pm bi$, the corresponding fundamental system contains the sequences

$$r^n \cos(n\theta), \quad \text{and } r^n \sin(n\theta)$$

where $r = \sqrt{a^2 + b^2}$ and θ is such that $a = r\cos\theta$ and $b = r\sin\theta$.

Exercise

- Find all solutions to

$$y(n+2) + 2y(n+1) + 2y(n) = 0.$$

Exercise: After winning the \$200 million jackpot at the Powerball lottery you are given the choice to either receive the lump sum of \$100 million or to receive \$10 million per year for the next 20 years. You go first to a financial advisor who tells you that you can invest our money with him and receive an interest rate of α percent yearly (compounded annually). To compare your two options and because you are an extremely thrifty individual you decide to invest all your money for the next 20 years. Write down difference equations for the two options. For which interest rate α is the option of lump sum better?

Further recurrence relations

Here we would like to recall that the general 'first order' difference equations are of the form

$$a_{n+1} = f(a_n), \ a_0 = a$$

for some a and f given. Here, we would like to note that these exercises were tackled at the beginning of the course as 'implicitly given sequences.

Exercise

- Find the sequence a_n satisfying

$$a_1 = \frac{1}{2}, \qquad a_{n+1} = \frac{(n+1)^2}{n(n+2)} a_n.$$

That's all folks!!!