UCT, Math, Exercise book

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Obsah

1 Logic, sets, mappings

1.1 Connections.

- *Complete the following table:*

A	B	C	$A \vee (\neg C)$	$(A\&B) \vee C$	$A \Rightarrow (B \Rightarrow C)$	$A \vee (B \Leftrightarrow C)$
true	*true*	*true*				
true	*true*	*false*				
true	*false*	*true*				
true	*false*	*false*				
false	*true*	*true*				
false	*true*	*false*				
false	*false*	*true*				
false	*false*	*false*				

- *Have three propositions: $A =$'Kutná Hora is the capital of Czechia', $B =$'Praha is the capital of Czechia', $C =$'two plus two is four' and $D =$'Pigs can fly'. Write down the following sentences and decide about their validity:*

  1. $A \vee B$.
  2. $A \Leftrightarrow D$.
  3. $A \Rightarrow C$.
  4. $C \Rightarrow A$.
  5. $B \vee D$.
  6. $B\&C$.
  7. $\neg A\&C$.

1.2 Quantifiers. *We define the following:*

- *a: Anastazia*
- *b: Bart*
- *c: Cicero*

- *$B(x,y)$: x belongs to y*
- *$D(x,y)$: x hates y*
- *$C(x)$: x is a cat*
- *$F(x)$: x is wild*
- *$P(x)$: x is a human.*

Try to rewrite the following formulas into sentences (try to make the sentences as nice as possible):

1. $C(b)\&F(b)\&B(b,c)$
2. $\forall x,\ (C(x) \Rightarrow D(a,x))$
3. $\exists x,\ (C(x)\&F(x)\&B(x,y))$
4. $\forall x, \forall y,\ ((C(x)\&F(x)) \Rightarrow (P(y) \Rightarrow D(y,x)))$
5. $\forall x,\ (C(x) \Rightarrow \exists y,\ (P(y)\&B(x,y)))$
6. $\neg\exists x,\ (C(x)\&B(x,a))\&\exists x,\ (F(x) \Rightarrow D(a,x))$.

1.3 Sets.

1. *Find* $\sup A$ *and* $\inf A$ *for* $A = \left\{\frac{p}{p+q},\ p,q \in \mathbb{N}\right\}$.
2. *Show that* $\sup[0,2] = \sup(0,2) = 2$.
3. *Let* $A, B \subset \mathbb{R}$ *be nonempty sets. Try to express* $\sup(A \cup B)$ *and* $\sup(A \cap B)$ *by* $\sup A$ *and* $\sup B$*, if it is possible.*

1.4 Math induction.

1. *Prove that for all* $n \in \mathbb{N}$ *it holds that*
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
2. *Prove that six is a divisor of* $n^3 + 5n$ *for every* $n \in \mathbb{N}$.
3. *Prove that*
$$(1+x)^n \geq 1 + nx$$
for every $x > -1$ *and every* $n \in \mathbb{N}$.
4. *Prove that*
$$1 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$$
for all $n \in \mathbb{N}$.
5. *Prove that* 6 *is a divisor of* $10^n - 4$ *for every* $n \in \mathbb{N}$.

1.5 Mappings.

1. *Which of these subsets are mappings?*
   - $f = \{\langle 1,5\rangle, \langle 2,4\rangle, \langle 1,3\rangle\}$,
   - $g = \{\langle 1,2\rangle, \langle 5,3\rangle, \langle 10,1\rangle\}$,
   - $h = \{\langle 3,3\rangle, \langle 4,3\rangle, \langle 7,7\rangle, \langle 10,3\rangle\}$.

   If f*,* g*, or* h *is a mapping, determine its domain and range.*
2. *Let* $A = \{1,2,3,4\}$ *and* $B = \{2,5,8\}$*. Consider a mapping* $f : A \to B$ *which is given as*
$$\{\langle 1,5\rangle,\ \langle 3,2\rangle,\ \langle 2,2\rangle\}.$$
Write down $\mathrm{Dom} f$*,* $\mathrm{Ran} f$ *and decide whether* f *is injection or surjection. Then let* $g : B \to A$ *be defined as* $g = \{\langle 2,1\rangle, \langle 8,4\rangle\}$*. Determine* $g \circ f$.
3. *Find* f^{-1} *for a function* $f(x) = \frac{x+3}{2x-1}$, $x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$.
4. *Show that* $f(x) = x + 2$, $x \in [3, \infty)$ *is bounded from below and not from above.*

2 Real functions

2.1 Basic properties.

1. *Find maximal domain of functions*

   - $f(x) = \sqrt{\frac{x+1}{x-1}}$,
   - $f(x) = \frac{1}{\sqrt{x^2+5x+4}}$,
   - $f(x) = \frac{1}{(\log(\sin x))^2}$.
   - $f(x) = \frac{\sqrt{x}}{e^x}$
   - $f(x) = \frac{1}{\ln x}$
   - $f(x) = \sqrt{\frac{x+1}{x-1}}$
   - $f(x) = \sqrt{x^2+6x+3} + \sqrt[3]{x+1}$
   - $f(x) = 5^{x^2+\ln x}$
   - $f(x) = \sqrt{\ln x} + \frac{1}{\sqrt{|x^2+4x+3|}}$

2. *Decide about the parity of the following functions*

   - $f(x) = \frac{\sin x}{x^3+x}$
   - $f(x) = \sqrt{x^2+1}\cos x$
   - $f(x) = \frac{x+1}{x-1}$
   - $f(x) = x^2 + x^4 + 3$
   - $f(x) = x^2 + \sqrt{x^2}$
   - $f(x) = x^3 + \frac{x}{x^2+1}$
   - $f(x) = \frac{x^2+1}{x}$
   - $f(x) = \frac{x+1}{x^2}$
   - $f(x) = x^2 \sin x$

   and justify your answer.

3. *Find f^{-1}:*

   (a) $f : y = 3x + 4$
   (b) $f : y = \frac{x}{x-3}$
   (c) $f : y = x^2 + 8x + 3,\ \mathrm{Dom} f = (-\infty, -4)$
   (d) $f : y = 3 + \frac{x}{x+1}$
   (e) $f : y = x^2 + 1$
   (f) $f : y = 4 + \frac{1}{x}$

4. *Sketch a graph of a function*
   $$f(x) = \mathrm{sgn}(\sin x)$$
   and decide about monotonicity, periodicity, range, domain, boundedness, and continuity. Here sgn *is defined as follows*
   $$\mathrm{sgn}(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ if } x < 0. \end{cases}$$

5. *Prove that the function f from the previous exercise is continuous in every point of a set $\mathbb{R} \setminus \{x = k\pi,\ k \in \mathbb{Z}\}$ and discontinuous everywhere else.*

2.2 Polynomials.

1. *Find all roots of $p(x) = x^2 + 5x + 6$.*
2. *Find all (complex) roots of $p(x) = 2x^2 + 8x + 16$.*
3. *Find all roots of $p(x) = x^3 + 3x^2 - 10x - 24$.*
4. *Find all roots of $p(x) = x^4 - 4x^3 + 12x^2 - 48x$.*

2.3 Real powers.

1. *Decide, which of the two numbers is higher*

   - $5^{1/4}$, $5^{1/2}$,
   - $\left(\frac{2}{3}\right)^2$, $\left(\frac{2}{3}\right)^{2.2}$,
   - $(\sqrt{2})^{-1}$, $(\sqrt{2})^{-0.66}$.

2. *Find all x satisfying*
$$\sqrt{x - x^2 + 12} < \sqrt{7 - 3x}.$$

2.4 Exponential functions and logarithms.

1. *Solve*
$$\frac{27^{3x-2}}{243} = 81^{3x-7}.$$
2. *Solve*
$$4^x + 2^{x+2} = 5.$$
3. *Solve*
$$\log_4(x^2 - 9) - \log_4(x + 3) = 3.$$
4. *Find all real numbers x satisfying*
$$e^x < c.$$

2.5 Goniometric functions.

1. *Solve* $\sin x = \frac{1}{2}$.
2. *Solve* $4\sin^2 x - 4\sin x + 1 = 0$.
3. *Find all $x \in \mathbb{R}$ satisfying*
$$\cos x > -\frac{1}{2}.$$

3 Sequences and their limits

3.1 Basics.

1. *Find an explicit formula for a sequence given as*
$$a_1 = 1, \quad a_{n+1} = (n+1)a_n$$
and verify your claim.
2. *Find an explicit formula for a sequence given as*
$$a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{n}{n+1}a_n$$
and verify your claim.
3. *Decide about the monotonicity of*
$$a_n = \frac{n}{(n^2+3)}$$
and verify your claim.
4. *Decide about the monotonicity of*
$$a_n = \frac{n+1}{n^2}$$
and verify your claim.

3.2 Limits.

1. *Solve*
$$\lim \frac{n^2+1}{(1-n)(n+2)}$$
2. *Solve*
$$\lim \frac{(n^2+2)^{3/2}}{n-1}$$

3. *Solve*

$$\lim 2 - \frac{2^n + 1}{3^n}$$

4. *Solve*

$$\lim \frac{1 - n^3}{n + 3}$$

5. *Solve*

$$\lim \left(\sqrt{2n} - \sqrt{2n-1}\right) \sqrt{n}$$

6. *Solve*

$$\lim \frac{\sqrt{n} + n^{\frac{1}{3}}}{n^{\frac{1}{2}} - 1}$$

7. *Solve*

$$\lim \frac{\sqrt{n^2 + 4n} - n}{5}$$

8. *Solve*

$$\lim \frac{(n+1)^4}{(n + \sqrt{n})^3}$$

9. *Solve*

$$\lim \left(\frac{2n-3}{2n}\right)^n$$

10. *Solve*

$$\lim \frac{\sqrt{n^2 + 2n + 2} - n}{\sqrt{n}}$$

11. *Solve*

$$\lim \frac{\sqrt[3]{n}^4 + n^2 2^n}{n - 1 + e^n}$$

12. *Solve*

$$\lim \frac{2 + n + 3^n}{2^n + n^2 - 1}$$

13. *Solve*

$$\lim \left(\frac{n^2 - 4}{n^2 + 5}\right)^{n^2}$$

14. *Solve*

$$\lim \frac{\sqrt{n+2}\sqrt{2n+5}}{4n - 3}$$

15. *Solve*

$$\lim \frac{\sqrt[3]{n^2}}{n+1} \sin n!$$

16. *Solve*

$$\lim \frac{(n+4)^8 - (n^2+1)^4}{(2n-4)^7}$$

17. *Solve*

$$\lim \frac{n + (-1)^n}{2n + 3}$$

18. *Solve*

$$\lim \frac{3n^6 + 4n - 3}{(2n+3)^6}$$

19. *Solve*

$$\lim \sqrt{n^4 - 5n} - n^2$$

20. *Solve*

$$\lim \frac{(-1)^n(\sqrt{n^2+1} - 1)}{n}$$

21. *Solve*

$$\lim \frac{\left(\frac{1}{2}\right)^{n^2} - \left(\frac{1}{3}\right)^{n^2}}{\left(\frac{1}{2}\right)^{n^2+1} - \left(\frac{1}{3}\right)^{n^2+1}}$$

22. *Solve*

$$\lim \left(\frac{2n-1}{2n+1}\right)^n$$

23. *Solve*

$$\lim \sqrt{n^2 + 3n} - n$$

24. *Solve*

$$\lim \frac{(2n+1)^2(n-2)^3}{(n+1)^5}$$

25. *Solve*

$$\lim (-1)^n \sqrt{n}(\sqrt{n+2} - \sqrt{n+1})$$

26. *Solve*

$$\lim \frac{\sqrt{n}(2n-4)}{\sqrt{n^3 + 3n}}$$

27. *Solve*

$$\lim \frac{(2n^2 - 4n + 1)\sqrt{n}}{\sqrt{n^5 + 4n}}$$

28. *Solve*

$$\lim \left(\frac{n}{n-2}\right)^n$$

29. *Solve*

$$\lim \sqrt[3]{n}^2 (\sqrt[3]{n} - \sqrt[3]{n-1})$$

30. *Solve*

$$\lim \frac{3^n + n2^n}{4^n}$$

31. *Solve*

$$\lim \frac{n^2 + 1}{(-1-n)(n+2)}$$

32. *Solve*

$$\lim \sqrt{n}(\sqrt{2n} - \sqrt{2n-1})$$

33. *Solve*

$$\lim \frac{1 - n^3}{n + 3}$$

34. *Solve*

$$\lim \sqrt{n^2 + 2n + 2} - n$$

35. *Solve*

$$\lim \frac{(n+1)^4}{(n + \sqrt{n})^3}$$

4 Functions II

4.1 Limits.

1. *Solve*
$$\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}$$
2. *Solve*
$$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}$$
3. *Solve*
$$\lim_{x\to\infty}\frac{(x+1)(x+2)(x+3)(x+4)(x+5)}{(5x+1)^5}$$
4. *Solve*
$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$$
5. *Solve*
$$\lim_{x\to\pi}\frac{\sin x}{x-\pi}$$
6. *Solve*
$$\lim_{x\to 0}\frac{5^x-e^x}{x}$$
7. *Solve*
$$\lim_{x\to 3} x+2$$
8. *Solve*
$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2+3x-18}$$
9. *Solve*
$$\lim_{x\to+\infty}\frac{1}{2}x^3-x+1$$
10. *Solve*
$$\lim_{x\to+\infty}\frac{x+2}{x^2+3}$$
11. *Solve*
$$\lim_{x\to-\infty}\frac{x^4}{x+1}$$
12. *Solve*
$$\lim_{x\to 0}\frac{x^3+x+1}{x^2+x}$$
13. *Solve*
$$\lim_{x\to -1}\frac{x^3-x+2}{x^2+2x+1}$$
14. *Solve*
$$\lim_{x\to+\infty}\frac{x^3+2x}{x^3-1}$$
15. *Solve*
$$\lim_{x\to-\infty}\sqrt{x^2+1}$$
16. *Solve*
$$\lim_{x\to 4}\frac{x^2-18}{x^2-8x+16}$$
17. *Solve*
$$\lim_{x\to 3}\frac{x^3-2x^2-2x-3}{x^2-6x+9}$$
18. *Solve*
$$\lim_{x\to-\infty}\frac{x^4+4x+1}{x^3-x+1}$$
19. *Solve*
$$\lim_{x\to 0} x\cot(3x)$$
20. *Solve the following limit of sequence*
$$\lim\left(1+\frac{1}{n}\right)^n$$
21. *Solve*
$$\lim_{x\to 0}(x+e^x)^{\frac{1}{x}}$$
22. *Solve the following limit of sequence*
$$\lim\left(\frac{2+3n}{3n-1}\right)^{2n}$$
23. *Solve*
$$\lim_{x\to 0}\frac{\sin 3x}{x}$$
24. *Solve*
$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}$$
25. *Solve*
$$\lim_{x\to\infty} e^{-x}$$
26. *Solve*
$$\lim_{x\to-\infty}\frac{\sin x}{x}$$
27. *Solve*
$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$$
28. *Solve*
$$\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{e^{2x}-e^{-2x}}$$
29. *Solve*
$$\lim_{x\to 0} x^x$$
30. *Solve*
$$\lim_{x\to 1}\frac{\ln x}{e^{x-1}-1}$$

4.2 Derivatives. *Compute:*

1. $\left(\frac{1}{3}x^4+x-1\right)'$
2. $\left((2x^2+3)^3\right)'$
3. $\left(\sqrt{x^2+4}\right)'$
4. $\left(x^2\sqrt{x+1}\right)'$
5. $\left(x^3+\sin x+e^{x^2}\right)'$
6. $\left(e^{x^2\sin x}\right)'$
7. $\left(3^{x^2+3x}\right)'$
8. $\left(x^3\frac{x^2+e^{\sin x}}{x^2+1}\right)'$
9. $\left(x^{x^2}\right)'$
10. $(\sin x\cdot\cos x)'$
11. $\left(\sqrt{\frac{x-1}{x+1}}\right)'$
12. $\left(x^2\cdot\sqrt{\frac{\sin x}{2+\sin x}}\right)'$
13. $(e^{2x})^{(10)}$
14. $(\cos x)^{(11)}$
15. $(x^{12}+5x^4)^{(6)}$
16. $(\ln x)^{(12)}$

4.3 Derivatives of higher order. *Compute*

1. $(\cos x)^{(11)}$
2. $(x^{12}+5x^4)^{(6)}$
3. $\left(e^{2x}\right)^{(10)}$
4. $(\log x)^{(12)}$

4.4 Monotonicity. *Find intervals where is the given function increasing or decreasing*

1. $f(x)=\frac{2x-1}{x+1}$
2. $f(x)=x^3-3x^2-9x+1$
3. $f(x)=\frac{x}{x^2+1}$
4. $f(x)=x^2-\frac{1}{x}$

4.5 Local extremes. *Find local extremes of the following functions*

1. $f(x)=\sqrt{2-x-x^2}$
2. $f(x)=x^4-\frac{4}{3}x^3-2x^2+4x+1$
3. $f(x)=e^{x-x^2}$
4. $f(x)=\sin(x^2)$

4.6 Inflexion. *Find intervals where is the given function convex or concave. Determine the inflexion points.*

1. $f(x)=x^4-12x^2+\sqrt{3}x.$
2. $f(x)=\frac{x}{x+1}.$
3. $f(x)=\frac{x}{\log x}.$
4. $f(x)=x\,(\log x)^2.$

4.7 Exercises with parameter.

1. *Find* $a,\ b\in\mathbb{R}$ *such that the point* $\langle 1,3\rangle$ *is a point of inflexion of* $f(x)=ax^3+bx^2.$

4.8 The course of a function. *Examine the course of* f*:*

1. $f(x)=3x-x^3$
2. $f(x)=\frac{e^x}{1+x}$
3. $f(x)=\frac{\sin x}{2+\cos x}$
4. $f(x)=\frac{x^4}{(1+x)^3}$
5. $f(x)=(x-3)\sqrt{x}$
6. $f(x)=(x-4)\sqrt[3]{x}$
7. $f(x)=3+\sin x\cos x$
8. $f(x)=\arctan\left(\frac{\sqrt{3}}{x^2}\right)$

4.9 l'Hospital rule. *Use the l'Hospital rule to solve*

1. $\lim_{x\to\infty}\frac{x^2}{e^x}$
2. $\lim_{x\to 0_+}x\log x$
3. $\lim_{x\to\pi/2}\frac{\tan(3x)}{\tan x}$
4. $\lim_{x\to 1_-}\log x\log(1-x)$
5. $\lim_{x\to 0}\frac{\tan x-x}{x-\sin x}$
6. $\lim_{x\to 0}\frac{\arctan x}{x^2-x}$

7. $\lim_{x\to 1} \frac{x^2+2x}{xe^x}$

8. $\lim_{x\to 0} \frac{x^3}{(1-\cos x)\sin x}$

4.10 Approximation. *Use the second-order Taylor polynomial to find an approximate value of*

1. $\sqrt[3]{30}$

2. $\log 1.2$

5 Integrals

5.1 Basics.

1. $\int x^3 - 3\sqrt{x}\ \mathrm{d}x$
2. $\int \left(\frac{1-x}{x}\right)^2\ \mathrm{d}x$
3. $\int \frac{x^2}{1+x^2}\ \mathrm{d}x$
4. $\int \frac{2^{x+1}-5^{x-1}}{10^x}\ \mathrm{d}x$
5. $\int \frac{e^{3x}+1}{e^x+1}\ \mathrm{d}x$
6. $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\ \mathrm{d}x$

5.2 Linear substitution. *Use a substitution $t = ax + b$ to solve*

1. $\int \left(\frac{x}{2}-3\right)^6\ \mathrm{d}x$
2. $\int 2\sin(3-2x)\ \mathrm{d}x$
3. $\int \frac{1}{x^2+4}\ \mathrm{d}x$
4. $\int \frac{3}{2x-4}\ \mathrm{d}x$
5. $\int \sqrt[3]{1-3x}\ \mathrm{d}x$
6. $\int \frac{2}{2x^2+4x+3}\ \mathrm{d}x$
7. $\int \tan(2x)\ \mathrm{d}x$
8. $\int e^{-x}+e^{-2x}\ \mathrm{d}x$

5.3 Substitution. *Use an appropriate substitution to solve the following integrals.*

1. $\int \frac{4x}{2x^2+1}\ \mathrm{d}x$
2. $\int \frac{x^3}{\sqrt{1+x^2}}\ \mathrm{d}x$
3. $\int \frac{1}{\sin^3 x}\ \mathrm{d}x$
4. $\int \frac{\sqrt{1+\log x}}{x\log x}\ \mathrm{d}x$
5. $\int \frac{\sin x\cos^3 x}{1+\cos^2 x}\ \mathrm{d}x$
6. $\int \cos x\tan^3 x\ \mathrm{d}x$
7. $\int \frac{e^x}{1+e^{2x}}\ \mathrm{d}x$
8. $\int x^2\sqrt[3]{1+x^3}\ \mathrm{d}x$

5.4 Integration by parts. *Use the integration by parts method to solve*

1. $\int x\cos x \mathrm{d}x$
2. $\int (x+1)e^{-x}\ \mathrm{d}x$
3. $\int \log x\ \mathrm{d}x$
4. $\int e^x\cos x\ \mathrm{d}x$
5. $\int (x^2+1)\sin x\ \mathrm{d}x$
6. $\int \sin^2 x\ \mathrm{d}x$
7. $\int x^2\sin 2x\ \mathrm{d}x$
8. $\int \sin 2x\cos x\ \mathrm{d}x$

5.5 Rational functions. *Solve*

1. $\int \frac{2x^4+5x^2-2}{2x^3-x-1}\ \mathrm{d}x$
2. $\int \frac{2x^3}{x+1}\ \mathrm{d}x$
3. $\int \frac{x^3+2x^2+x-1}{x^2-x+1}\ \mathrm{d}x$
4. $\int \frac{7x^2+7x-6}{x^2-3x}\ \mathrm{d}x$
5. $\int \frac{x}{(x+1)(x+2)(x-3)}\ \mathrm{d}x$
6. $\int \frac{3x+6}{x^2+6x+13}\ \mathrm{d}x$

5.6 Mixture. *Solve (using an appropriate method)*

1. $\int x^5e^{x^3}\ \mathrm{d}x$
2. $\int \arctan x\ \mathrm{d}x$
3. $\int 2x\sin(x^2)\ \mathrm{d}x$
4. $\int \frac{\sin x\cos x}{(1-\cos x)(2+\cos x)}\ \mathrm{d}x$
5. $\int \arctan\sqrt{x}\ \mathrm{d}x$
6. $\int \frac{1}{\sin x\cos^2 x}\ \mathrm{d}x$

5.7 Riemann's integral. *Compute the following Riemann integrals by definition*

1. $\int_{-1}^{1} x^2 \;\mathrm{d}x$

2. $\int_0^1 a^x \;\mathrm{d}x$, $a > 1$ *given.*

5.8 Newton's integral. *Compute*

1. $\int_0^4 \frac{x^2}{3} + x + 1 \;\mathrm{d}x$
2. $\int_0 -^{\pi} \frac{1}{2}\sin x - \cos x \;\mathrm{d}x$
3. $\int_2^5 x^2 e^x \;\mathrm{d}x$
4. $\int_1^3 x\sqrt{x^2+4} \;\mathrm{d}x$
5. $\int_{-2}^{5} \frac{1}{3x+5} \;\mathrm{d}x$
6. $\int_0^1 \frac{\arctan x}{x^2+1} \;\mathrm{d}x$
7. $\int_{\pi/4}^{\pi/2} \cos^2 x \;\mathrm{d}x$
8. $\int_{\pi/2}^{3\pi/2} \sin^4 x \cos x \;\mathrm{d}x$

5.9 Areas.

1. *Compute the area of a triangle M whose sides are on lines $y = 3 - x$, $y = 2x$ and $y = \frac{1}{2}x$.*
2. *Compute the area of $M = \{\langle x, y\rangle \in \mathbb{R}^2, x^2 + 2x - 8 < y < x - 2\}$.*
3. *Compute the area of $M = \{\langle x, y\rangle \in \mathbb{R}^2, x \in (0, \frac{7}{4}\pi),\ \sin x < y < \cos x\}$.*
4. *Compute the area of $M = \{\langle x, y\rangle \in \mathbb{R}^2, x^2 + y^2 < R^2\}$ where $R > 0$ is given. Hint: use substitution $x = R\sin t$.*

5.10 Curve length. *Compute the length of the graph of a given function f:*

1. $f(x) = x^{3/2}$, $x \in (0, 1)$
2. $f(x) = \frac{e^x + e^{-x}}{2}$, $x \in (-1, 1)$
3. $f(x) = \log x,\ x \in (\sqrt{3}, \sqrt{8})$
4. $f(x) = \log(\sin x),\ x \in \left(\frac{\pi}{3}, \frac{2\pi}{8}\right)$

5.11 Volumes.

1. *Compute the volume of the solid arising by the rotation of*
$$M = \{\langle x, y\rangle \in \mathbb{R}^2,\ 0 \le y < x, x \in (0, 2)\}$$
around the axis y.
2. *Compute the volume of the solid arising by the rotation of*
$$M = \{\langle x, y\rangle \in \mathbb{R}^2,\ x^2 < y < 2 - x^2\}$$
around the axis x.

5.12 Improper integrals. *Decide about the convergence of the following integrals*

1. $\int_0^\infty \frac{x^2}{x^4 - x^2 + 1} \;\mathrm{d}x$
2. $\int_{-1}^{1} \frac{\sin x}{(x^2-1)(x-1)} \;\mathrm{d}x$
3. $\int_0^\infty \frac{1}{\sqrt{x^3 + x}} \;\mathrm{d}x$
4. $\int_0^\infty \frac{\arctan x}{\sqrt{x}^3} \;\mathrm{d}x$

6 Ordinary differential equations

6.1 General questions.

1. *Show that $y(x) = e^x - x$ is a solution to*
$$y' + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1.$$
2. *Show that $y(x) = x^2 - x^{-1}$ is a solutions to*
$$x^2 y'' = 2y$$
on an interval $(0, \infty)$.
3. *Decide, whether is the following equation linear or nonlinear and determine its order*
$$5y'' + 4y' + 9y = 2\cos(3x).$$
4. *Decide, whether is the following equation linear or nonlinear and determine its order*
$$y' = \frac{y(2 - 3x)}{x(1 - 3y)}.$$
Can be the equation solved by the separation of variables?
5. *Decide, whether is the following equation linear or nonlinear and determine its order*
$$y\left(1 + (y')^2\right) = 5.$$
Can be the equation solved by the method of separation of variables?
6. *Determine the right hand side and the appropriate homogeneous problem of the following equation*
$$y'' + 3y' + 7 = 0.$$

6.2 Separation of variables. *Find all solutions to*

1. $y' = 1 - y^2$
2. $y' = \sqrt{1-y^2}$
3. $y' = y \log y \sin x$
4. $y' = \frac{2y^2 - xy}{x^2}$

6.3 Linear equations of the first order. *Find all solutions to*

1. $x^2 y' - xy = 1$
2. $xy' + (1+x)y = e^x$
3. $y' + \frac{x}{1+x^2} y = \frac{1}{x(x^2+1)}$
4. $y' - 2y = e^{4x}$, *then find the particular solution fulfilling* $y(0) = 2$
5. $y' - \frac{1}{x} y = x^2 e^x$

6.4 Linear equations with constant coefficients. *Find all solutions to*

1. $y^{(4)} + 18y'' + 81y = 0$
2. $y''' - 9y'' = 0$
3. $y''' + 3y'' + 3y' + y = 0$
4. $y'' + 6y' + 9y = 0$, *then find the particular solution fulfilling* $y(0) = 2$, $y'(0) = 1$
5. $y'' + 2y' + 5y = 0$, *Then find the particular solution fulfilling* $y(0) = 0$, $y'(0) = 1$
6. $y'' - y' + y = \cos x - \sin x$
7. $y''' + y'' = x^3 + x^2$
8. $y''' + y'' + y' + y = x^3 + x^2 + x + 1$
9. $y''' - 3y'' + 4y = e^{2x}$
10. $y'' - 2y' + y = \frac{e^x}{x}$
11. $y'' - y' = \frac{x+1}{x^2}$

6.5 Applications.

1. *If* $P(t)$ *is the amount of dollars in savings bank account that pazs a yearly interest rate of* $r\%$ compounded continuously, *then*
$$P' = \frac{r}{100} P,$$
where t *is in years. Assume* $r = 5$ *and* $P(0) = 1000$ *USD.*
   (a) *How much will be in the account after two years?*
   (b) *When will the account reach* 4000 *USD?*
   (c) *If* 1000 *USD is added every 12 months, how much will be in the account after three and half years?*
2. *The logistic equation for the population* p *at time* t *of a certain species is given by*
$$p' = p(2000 - p).$$
   (a) *If the initial population is* 3000, *what can you say about the limiting population* $\lim_{t\to\infty} p(t)$*?*
   (b) *Can a population of* 1000 *ever decline to* 500*?*
   (c) *Can a population of* 1000 *ever increase to* 3000*?*
3. *In 1790 the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Using the Malthusian model ($p'(t) = kp(t)$), estimate the U.S. population as a function of time.*
4. *Use the logistic model ($p'(t) = kp(t)(K - p(t))$) to estimate the U. S. population if you now (in addition) that in 1840 there was a population of 17.07 million.*

6.6 Difference equations – linear case. *Find all sequences* y_n *solving*

1. $y_{n+3} + y_{n+2} - 8y_{n+1} - 12y_n = 0$
2. $y_{n+2} - 5y_{n+1} + 6y_n = 1 + n$
3. $y_{n+2} - 5y_{n+1} + 4y_n = 4^n - n^2$

6.7 Difference equations – recurrence relations. *Find the exact formula of a sequence* a_n *which is given as*

1. $a_{n+1} = a_n + \frac{1}{(n+2)(n+1)}$, $a_1 = \frac{1}{2}$
2. $a_{n+1} = \frac{n}{n+2} a_n$, $a_1 = \frac{1}{2}$

7 Series

7.1 Particular values. *Try to evaluate the following sums*

1. $\sum_{n=4}^{\infty} \frac{4}{3^n}$
2. $\sum_{n=1}^{\infty} \frac{2^{n+1}+5^{n-1}}{10^n}$
3. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
4. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

7.2 Convergence. *Decide about the convergence of the following series*

1. $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$
2. $\sum_{n=1}^{\infty} \frac{n2^n}{3^n}$
3. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n^n}$
4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}\sqrt{2n+1}}$
5. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$
6. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{\frac{1}{n}}$
7. $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{6^n}$
8. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$
9. $\sum_{n=1}^{\infty} \frac{3^n}{n^2 5^n}$
10. $\sum_{n=1}^{\infty} (-1)^{n(n-1)} \frac{n!}{(2n)!}$
11. $\sum_{n=1}^{\infty} \left(e^{\frac{1}{n^2}} - 1\right)$