UCT, Math, Exercise book

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Obsah

1 Linear algebra

1.1 Vector spaces

1. Let $u = (1,2,3)$, $v = (-3,1,-2)$ and $w = (2,-3,-1)$. Compute $u+v$, $u+v+w$, $2u+2v+w$.

2. Let $u = (4,3,2)$, $v = (1,3,5)$, $w = (3,6,9)$. Are u,v,w linearly dependent or linearly independent?

3. Determine, whether a vector $z = (1,2,1)$ belongs to span$\{u,v\}$ where $u = (1,1,-1)$ and $v = (1,2,1)$.

4. Find all α such that $u = (\alpha,1,0)$, $v = (1,\alpha,1)$ and $w = (0,1,\alpha)$ are linearly dependent.

5. We define operations $\oplus$ and $\odot$ on $\mathbb{R}^2$ such that
$$(x_1,y_1)\oplus(x_2,y_2) = (x_1+x_2,0),\ \alpha\odot(x_1,y_1) = (\alpha x_1,\alpha y_1)$$
for all $\alpha \in \mathbb{R}$ and $(x_1,x_2) \in \mathbb{R}^2$. Is $\mathbb{R}^2$ equipped with these operations a vector space? Justify your answer.

6. Can be $x = (1,3,2,-1)$ written as a linear combination of $u = (1,1,0,-2)$, $v = (0,2,-1,1)$ and $w = (-2,1,1,0)$?

7. Does a polynomial $P(x) = x^2+4x+5$ belong to the linear span of $Q(x) = x^2+2$ and $R(x) = x-3$?

8. Do all solutions to

$$x + 3y - 2 = 0$$

form a linear space? How about all solutions to

$$x + 3y = 0.$$

Verify your claim.

1.2 Matrices

1. Let $A = (1 \ \ -2)$ and $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Compute AB and BA.

2. Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ and $B = (2\ 2\ {-1})$. Compute AB^T.

3. Use a definition of rank$A$ to determine the rank of

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 \\ 2 & 4 & 2 & -2 \end{pmatrix}.$$

4. Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & 0 & 5 \end{pmatrix}$. Compute AB and BA (if they exist).

5. Compute

$$\begin{pmatrix} 1 & 0 & 4 \\ -1 & 1 & 0 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 2 & 5 \\ 1 & 1 & 1 \\ -0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 5 & -2 & -3 \\ 3 & 3 & -3 \\ 1 & 1 & 6 \end{pmatrix}$$

1.3 The Gauss elimination

1. Solve

$$\begin{aligned} z + 3x &= y + 6 \\ x &= y + z \\ 2x - 3y &= 7 - z \end{aligned}$$

2. Solve

$$\begin{aligned} px + y - z &= 0 \\ x + (p-1)y + z &= 3 \\ x + 2y + z &= p \end{aligned}$$

for all real parameters p.

3. Find A^{-1} for

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then use it to find solution to

$$\begin{aligned} 2x + y + z &= 3 \\ x + 3z &= -7 \\ 2x + y &= 1 \end{aligned}$$

and a solution to

$$\begin{aligned} 2x + y + z &= 0 \\ x + 3z &= 3 \\ 2x + y &= -1 \end{aligned}$$

4. Determine rank of a matrix $A = \begin{pmatrix} 3 & 2 & 3 \\ -2 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix}$.

5. Find all solutions to

$$\begin{aligned} x + 2y - 3z + 2t &= 4 \\ 2x - y - z - t &= -2 \\ 5x - 5z &= 0 \\ -5y + 5z - 5t &= -5 \end{aligned}$$

6. Find an inverse matrix to

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

1.4 Square matrices

1. Use the definition of determinant to compute

$$\det \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 5 \\ 2 & 1 & 4 & 4 \end{pmatrix}$$

2. Find the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 5 & 7 & 10 \\ 1 & 2 & 3 & 6 & 7 \\ 1 & 1 & 3 & 5 & 5 \\ 1 & 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

3. Use the Cramer rule to solve

$$\begin{aligned} x + 2y - z &= 0 \\ -x + y + 2z &= 1 \\ 2x + y - z &= 1. \end{aligned}$$

4. Compute

$$\det \begin{pmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 10 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

5. Find a matrix X such that

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

6. Find a matrix X such that

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

1.5 Eigenvalues

1. Compute all eigenvalues of $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ and find the corresponding eigenvectors.

2. Find all eigenvalues of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and find the corresponding eigenvectors.

3. A population of rabbits has the following characteristics:

   (a) Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3.

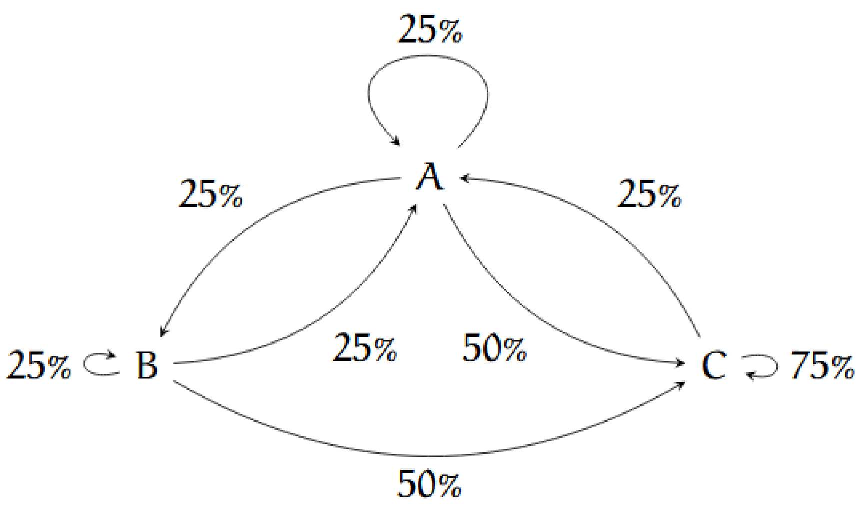
   (b) During the first year, the rabbits produce no offspring. The average number of offspring per parent is 6 during the second year and 8 during the third year.

   The population now consists of 24 rabbits in the first age, 24 rabbits in the second and 20 rabbits in the third. How many rabbits will there be in each age class in 1 year? Find a stable age distribution for the population of rabbits.

4. A rental car company has three basis in three different cities A, B, C. The customers can pick up and return cars at any station. For example, a customer could pick up his car at A and return it in C.
   We assume that there is only one car type available and that all customers rent their cars for one fixed time period (say one day). The graph below shows the proportions of cars that remain at a station or are transferred to other station over a day.

   (a) Find an appropriate matrix $P \in \mathbb{R}^3$ that describes this transition process, i.e., a matrix P such that $P\nu_0$ gives the distribution of cars after one day assuming ν_0 is the initial distribution.

   (b) How should the cars be distributed initially such that the distribution does not change over time? (i.e., find a distribution ν such that $P\nu = \nu$). Is it possible?



2 Geometry

2.1 Euclidean space

1. Show that $|\|v\| - \|w\|| \le \|v - w\|$ for all v, w from the Euclidean space.
2. Compute $(5, 1, -2) \times (4, -4, 3)$.
3. Find the angle between the vectors $(1, 0, -1, 3)$ and $(1, \sqrt{3}, 3, -3)$.
4. Which of the angles (if any) of triangle ABC, with $A = (1, -2, 0)$, $B = (2, 1, -2)$, and $C = (6, -1, -3)$, is a right angle?
5. Let $A = (5, 4)$ and $C = (2, 1)$ be points in the Euclidean space $\mathbb{R}^2$. Find a point B such that ABC is a right triangle with hypotenuse AB (i.e., $\sphericalangle ACB$ is a right angle) and such that $\varrho(A, C) = \varrho(A, B)$. Find both solutions.
6. Verify that $u \times v$ is perpendicular to both u and v.

2.2 Lines and planes

1. Let p be a line perpendicular to $q : \begin{cases} x = 3 - t \\ y = -2 + 2t \end{cases}$ containing a point $M = (-1, 1)$. Write p in a normal form.
2. Does a point $A = (1, 3)$ belong to a line segment BC where $B = (2, 4)$ and $C = (4, 6)$? What is the center of line segment AB?
3. Find a parametric equation (i.e., an equation of the form $X = A + tu + sv$, $t, s \in \mathbb{R}$) of a plane
$$\sigma : 3x - 2y + 6z - 14 = 0.$$
4. Find an intersection of lines p and q where $p : \begin{cases} x = -3 + t \\ y = -3 + 2t \\ z = 4 - 2t \end{cases}$ $t \in \mathbb{R}$ and $q : \begin{cases} x = 4 \\ y = 2 - 5t \\ z = 11t \end{cases}$ $t \in \mathbb{R}$. Then determine the angle between these two lines.
5. There are given points $A = (1, 0, 1)$, $B = (-1, 1, 2)$, $C = (0, 3, 1)$, $D = (-1, -1, -1)$, $E = (0, 2, 3)$.
   (a) Write a parametric equation of the plane σ which contains points A, B and C.
   (b) Write a parametric equation of the line p which contains points D and E.
   (c) Find an intersection of σ and p.

2.3 Topology

1. Show that $B_1(0, 0)$ is an open set.
2. Find $\partial\Omega$ for
$$\Omega = \{\langle x, y\rangle \in \mathbb{R}^2,\ x^2 + y^2 > 1,\ |y| \le x\}.$$
3. Determine, whether a set
$$M = \{\langle x, y\rangle \in \mathbb{R}^2, |x| \ge -2, |y| < 1,\ x^2 < 3\}$$
is open, closed or none of this. Justify your answer.

3 Functions of two variables

3.1 Introduction

1. Determine and sketch a domain of a function $f = x + \sqrt{y}$.
2. Determine and sketch a domain of a function $f = \sqrt[3]{\frac{1}{xy}}$.
3. Determine and sketch a domain of a function $f(x, y) = \sqrt{|x| + |y| - 2}$.
4. Determine and sketch a domain of a function $f(x, y) = \log(x \log(y - x))$.
5. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x, y) = xy$.
6. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x, y) = |x| + y$.
7. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x, y) = (x + y)^2$.
8. Find $g(t) = f(t, t + 1)$ where $f(x, y) = x^2 + 2xy$.
9. Find $g(t) = f(t, t^2)$ where $f(x, y) = x - y^2$.

3.2 Limits

1. Compute $\lim_{(x,y)\to(0,0)} \frac{x+y}{x+y+1}$.
2. Compute $\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$.
3. Compute $\lim_{x\to\infty} \left(\lim_{y\to\infty} \frac{x+y}{x^2+y}\right)$ and $\lim_{y\to\infty} \left(\lim_{x\to\infty} \frac{x+y}{x^2+y}\right)$.
4. Determine, whether a function
$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} \text{ for } (x,y) \neq (0,0) \\ 0 \text{ for } (x,y) = (0,0). \end{cases}$$
is continuous.
5. * Examine a limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.
6. Examine $\lim_{(x,y)\to(0,0)} \frac{2xy}{xy+2x-y}$.
7. Compute $\lim_{(x,y)\to(1,2)} \frac{x^3y-xy^3+1}{(x-y)^2}$.
8. Examine $\lim_{(x,y)\to(2,2)} \frac{x^3-y^3}{x^4-y^4}$.
9. Compute $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$.
10. Compute $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$.

3.3 Derivatives

1. Compute ∇x^y.
2. Compute $\nabla x \sin(x+y)$.
3. Compute all partial derivatives of the first and second order of a function $f = xy + \frac{x}{y}$.
4. Compute all second derivatives of $f(x,y) = \arctan \frac{x+y}{1-xy}$.
5. Compute the first and the second order partial derivatives of a function $f(x,y) = \frac{y}{\sin x}$.
6. Determine all first order derivatives of $f(x,y) = (3x+2y)^{\log y}$.
7. Determine an angle between gradients of functions $f(x,y) = \sqrt{x^2+y^2}$ and $g(x,y) = x - 3y + \sqrt{3xy}$ in the point $(3,4)$.
8. Find a point where $\nabla \log(x + \frac{1}{y})$ equals $\left(1, \frac{16}{9}\right)$.
9. Determine the derivative of $f(x,y) = 3x^4 + xy + y^3$ in the point $M = (1,2)$ in the direction of vector $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
10. Determine the derivative of $f(x,y) = x^3 - 2x^2y + xy^2 + 1$ in the point $M = (1,2)$ in the direction of vector $(3,4)$.
11. Determine the differential of a function $f(x,y) = x^2 - 2xy - 3y^2$ at the point $A = (-1,1)$.
12. Determine the tangent plane to $f(x,y) = x^4 + 2x^2y - xy + x$ in the point $A = (1,1,?)$.

3.4 Taylor polynomial

1. Write the second degree Taylor polynomial at point $(1,3)$ of a function $f(x,y) = x^2y + y^2$.
2. Write the second degree Taylor polynomial at point $(1,-2)$ of $f(x,y) = \sqrt{2x-y}$.
3. Write the second degree Taylor polynomial at point $(0,0)$ for $f(x,y) = \log(x^2+y^2+1)$.
4. Use the second degree Taylor polynomial to find an approximate value of $\sqrt{20 - (1.9)^2 - 6.6}$.

3.5 Implicit function theorem

1. Is there a function $y(x)$ determined by
$$x^2 + 2xy - y^2 = 2$$
in the neighborhood of $(1,1)$? If yes, compute $y'(1)$ and $y''(1)$.
2. Is there a function $y(x)$ determined by
$$x^y = y^x$$
on the neighborhood of $(2,2)$? If yes, compute $y'(2)$ and $y''(2)$./
3. Find a tangent line to a curve
$$e^{xy} + \sin y + y^2 = 1$$
passing through $(2,0)$.
4. Is there a function $y(x)$ determined by
$$y + xy^2 - xe^x = 0$$
in the neighborhood of $(0,0)$? If yes, write its approximation by the second order Taylor polynomial.
5. Is there a function $y(x)$ determined by
$$x^3 - 3xe^y - \sin(xy) = 0$$
on the neighborhood of point $(1,0)$? If yes, find out whether the point $x = 1$ is a local maximum or minimum of y.
6. Is there a function $y(x)$ determined by
$$x + y - e^{x-y} = 0$$
on the neighborhood of point $\left(\frac{1}{2}, \frac{1}{2}\right)$? If yes, write an equation of the tangent line passing through that point.

3.6 Extremes

1. Find local extremes of a function
$$f(x,y) = x^3 + 8y^3 - 6xy + 5.$$
2. Find a local maxima and minima of a a function
$$f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$$
3. Find all local maxima and minima of $f(x,y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$.
4. Find all local maxima and minima of $f(x,y) = y^3 + 3x^2y - 15y - 12x$.
5. Find the global maximum and minimum values of $f(x,y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.
6. Find the global maximum and minimum values of $f(x,y) = x^2 + 2y^2 - 4y$ subject to the constraint $x^2 + y^2 = 9$.
7. Find the global maximum and minimum values of $f(x,y) = x^2 + xy$ subject to the constraint $y \le 9$, $y \ge x^2$.
8. Find the maximum and the minimum values of $f(x,y) = x^2 + y^2$ on the set $M = \{(x,y) \in \mathbb{R}^2, xy \le 1, x \ge \frac{1}{2}, y \ge \frac{1}{2}\}$.
9. A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y - x)$ of the standard items and $45\ 000 + 500(x - 2y)$ of the deluxe each year. How should the items be priced to maximize the profit?
10. Find the maximum and minimum values of $f(x,y) = x^2 + y^2$ on the set $M = \{(x,y) \in \mathbb{R}^2,\ xy \le 1,\ x \ge 1/2,\ y \ge 1/2\}$.
11. Find the maximum and minimum of $f(x,y,z) = xyz$ on a set $M = \{x^2+y^2+z^2 = 1,\ x+y+z = 0\}$.
12. Find the maximum and minimum of $f(x,y,z) = z + e^{xy}$ on a set $M = \{x^2+y^2+z^2 = 1,\ x^2+y^2 = z\}$.
13. Find the maximum and minimum of $f(x,y,z) = x^2 + 2xz + y^2 + z$ on a set $M = \{x^2 + y^2 + z^2 \le 1,\ x = y^2 + z^2\}$.

3.7 Integrals

1. Change the order of integration of
$$\int_0^2 \left(\int_x^{2x} f(x,y)\ \mathrm{d}y \right) \mathrm{d}x.$$
2. Change the order of integration of
$$\int_0^1 \left(\int_{x^3}^{x^2} f(x,y)\ \mathrm{d}y \right)\ \mathrm{d}x.$$
3. Determine M_x and M_y where
$$M = \{y \ge x^2,\ y \le (x-2)^2,\ y \le (x+2)^2\}$$
4. Sketch M given as
$$M = \{x \ge 0,\ x \le y \le \sqrt{x}\}$$
and determine M_x and M_y.
5. Compute
$$\int_M 1\ \mathrm{d}x\mathrm{d}y$$
where
$$M = \{x^2 \le y \le 4 - x^2\}.$$
6. Compute
$$\int_M \frac{x+2}{x^2+1} y\ \mathrm{d}x\mathrm{d}y$$
where
$$M = \{2 \le x \le 3,\ 0 \le y \le 2\}.$$
7. Compute
$$\int_M x^2 + y\ \mathrm{d}x\mathrm{d}y$$
where M is a triangle with vertices $(0,0)$, $(3,1)$ and $(1,3)$.
8. Compute
$$\int_M xy^2\ \mathrm{d}x\mathrm{d}y$$
where
$$M = \{x^2 \le y \le x\}.$$
9. Evaluate
$$\int_M \frac{y^2}{x}\ \mathrm{d}x\mathrm{d}y$$
where M is the region between the parabolas $x = 1 - y^2$ and $x = 3(1 - y^2)$. Use the change of variables
$$x = v(1-u^2)$$
$$y = u.$$
10. Compute
$$\int_M (3x - 2y)\ \mathrm{d}x\mathrm{d}y,$$
where
$$M = \left\{ \frac{3}{2}x - 4 \le y \le \frac{3}{2}x + 2,\ -2x + 1 \le y \le -2x + 3 \right\}.$$

Use the change of variables

$$x = u + 2v$$
$$y = -2u + 3v.$$

11. Compute
$$\int_M \frac{4}{(x-y)^2}\, \mathrm{d}y\mathrm{d}x$$
where M is the trapezoid bounded by the lines $x - y = 2$, $x - y = 4$, $x = 0$, $y = 0$.

12. Let $H = \{(r, \alpha),\ 1 \le r \le 3,\ 0 \le \alpha \le \pi\}$. Consider $\Phi(r, \alpha) = (r\cos\alpha, r\sin\alpha)$. Sketch the set $\Phi(H)$.

13. Compute
$$\int_M 2xy\, \mathrm{d}x\mathrm{d}y$$
where M is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

14. Evaluate the integral
$$\int_M 3x\, \mathrm{d}x\mathrm{d}y$$
where
$$M = \{(x, y),\ 1 \le x^2 + y^2 \le 4,\ y \ge 0\}.$$

15. Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

16. Compute
$$\int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{0} \cos(x^2 + y^2)\, \mathrm{d}y \right) \mathrm{d}x.$$

4 Systems of ODEs

4.1 Basics

1. Write the system
$$w^{(4)} + w = t^2$$
as a first order system. Express the system in the matrix form $\mathbf{x}' = A\mathbf{x} + \mathbf{b}(t)$.

2. Rewrite the equation
$$x'''(t) - 2x''(t) + x'(t) + 2x(t) = t$$
into a system of first-order linear differential equations

3. Find a fundamental solution set for a system
$$\mathbf{x}'(t) = A\mathbf{x}$$
where
$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

4. Find a fundamental solution set for
$$\mathbf{x}'(t) = A\mathbf{x}(t), \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}.$$

5. Find a general solution to
$$\mathbf{x}'(t) = A\mathbf{x}(t), \text{ where } A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}.$$

6. Find the general solution to
$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}(t).$$

7. Find the fundamental solution system
$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{x}(t).$$

8. Find all solutions to
$$\mathbf{x}'(t) = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} \mathbf{x}(t).$$

9. Find a general solution to
$$\mathbf{x}'(t) = \begin{pmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{pmatrix} \mathbf{x}(t).$$

10. Find all solutions to
$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \mathbf{x}(t).$$

11. Find all solutions to
$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x}(t).$$

4.2 Undetermined coefficients

1. Solve
$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -t-1 \\ -4t-2 \end{pmatrix}.$$

2. **Conventional Combat Model**: A simplistic model of a pair of conventional forces in combat yields the following system:
$$\mathbf{x}'(t) = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}\mathbf{x} + \begin{pmatrix} p \\ q \end{pmatrix}$$
where x_1 and x_2 represent the strength of opposing forces at time t. The terms $-ax_1$ and $-dx_2$ represent the operational loss rates and the terms $-bx_2$ and $-cx_1$ represent the combat loss rates for the troops x_1 and x_2 respectively. The constants p and q represent the respective rates of reinforcements. Let $a = 1$, $b = 4$, $c = 3$, $d = 2$ and $p = q = 5$. By solving the appropriate initial value problem, determine which forces will win if

   (a) $\mathbf{x} = (20, 20)$.

   (b) $\mathbf{x} = (21, 20)$.

   (c) $\mathbf{x} = (20, 21)$.

3. Find a solution to
$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}.$$
satisfying $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

4.3 Variation of constants

1. Find the general solution to
$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} 8\sin t \\ 0 \end{pmatrix}.$$

2. Find a solution to
$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}.$$
satisfying
$$\mathbf{x}(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

4.4 Phase plane and stability

1. Find the critical points of the system
$$x' = x^2 - 2xy, \quad y' = 3xy - y^2.$$

2. Verify, that $x(t) = t + 1$ and $y(t) = t^3 + 3t^2 + 3t$ is a solution to
$$x' = 1, \quad y' = 3x^2.$$
Then sketch this particular trajectory.

3. Sketch the phase portrait of the equation
$$x' = (y - x)(y - 1), \quad y' = (x - y)(x - 1).$$

4. Find the critical points and solve the related phase plane differential equation for the system
$$x' = (x - 1)(y - 1), \quad y' = y(y - 1).$$
Describe the asymptotic behavior of trajectories that starts at $(3, 2)$, $(2, 1/2)$ and $(3, -2)$.

5. Classify the critical point at the origin and sketch the phase plane diagram for the system
$$\begin{aligned} x' &= x - 4y \\ y' &= 4x + y. \end{aligned}$$

6. Classify the critical point at the origin and sketch the phase plane diagram for the system
$$\begin{aligned} x' &= 5x - 3y \\ y' &= 4x - 3y. \end{aligned}$$

5 Introduction to differential geometry

5.1 Integrals of the first type

1. Compute
$$\int_{\mathcal{K}} \mathrm{d}s, \ \mathcal{K} \text{ is a line segment } AB, \ A = (-2, -4), \ B = (6, 12).$$

2. Compute the length of one turn of a helix, i.e., a length of a curve $\mathcal{K}$ which is parametrized by
$$r(t) = (a\sin t, a\cos t, bt), \ t \in [0, 2\pi].$$

3. Compute
$$\int_{\mathcal{K}} (x + z) \ \mathrm{d}s,$$
where $\mathcal{K}$ is a line segment AB, $A = (1, 2, 3)$, $B = (3, 2, 1)$.

4. Compute
$$\int_{\mathcal{K}} \frac{z^2}{x^2 + y^2} \ \mathrm{d}s$$

where $\mathcal{K}$ is parametrized by

$$r(t) = (\cos t, \sin t, t).$$

5. Compute

$$\int_{\mathcal{K}} \sqrt{(x-3)^2 + y^2}\ \mathrm{d}s$$

where $\mathcal{K} = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 + 9 = 6(x+y)\}$.

5.2 Integrals of the second type, potentials

1. Compute

$$\int_{\mathcal{K}} \left(\frac{x}{(x^2+y^2)^3}, \frac{y}{(x^2+y^2)^3} \right)\ \mathrm{d}r$$

where $r(t) = (4\cos t, 4\sin t),\ t \in (0, \pi/2)$.

2. Compute

$$\int_{\mathcal{K}} (x^2 - 2xy)\ \mathrm{d}x + (y^2 - 2xy)\ \mathrm{d}y$$

where $\mathcal{K}$ is a parabola $y = x^2$, $x \in [-1,1]$ with the initial point $A = (-1,1)$ and the terminal point $B = (1,1)$.

3. Compute

$$\int_{\mathcal{K}} (x+y, y+z, z+x)\ \mathrm{d}r$$

where $\mathcal{K}$ is parametrized by $r(t) = \left(10\cos t, 5\sin t, \frac{t^2}{10}\right)$ for $t \in [0, 20]$

4. Compute

$$\int_{\mathcal{K}} (e^x + y, xy^2)\ \mathrm{d}r$$

where $\mathcal{K}$ is a curve containing points $\{(x,y) \in \mathbb{R}^2, y^2 = x,\ x \le 3\}$ whose initial point is $(3, -\sqrt{3})$ and terminal point is $(3, \sqrt{3})$.

5. Compute

$$\int_{\mathcal{K}} (x^2 + y^2)\ \mathrm{d}x + (x^2 - y^2)\ \mathrm{d}y$$

where $\mathcal{K}$ is given by a parametrization $r(t) = (t, 1 - |1-t|)$, $t \in [0,2]$.

6. Find a potential of a vector field

$$F(x,y) = (e^y, xe^y - 2y).$$

7. Does a vector field

$$F = \left(3x^2y, x^3 + \sqrt{y}\right)$$

have a potential? If yes, compute it.

8. Consider a differential form

$$F(x,y,z) = (x^2+yz)\ \mathrm{d}x + (y^2+xz)\ \mathrm{d}y + (z^2+xy)\ \mathrm{d}z.$$

Does it have a potential? If yes, determine it.