1 Linear algebra

1.1 Vector spaces

1. Let $u = (1, 2, 3)$, $v = (-3, 1, -2)$ and $w = (2, -3, -1)$. Compute $u + v$, $u + v + w$, $2u + 2v + w$.

2. We define operations $\oplus$ and $\odot$ on $\mathbb{R}^2$ such that
$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, 0),\ \alpha \odot (x_1, y_1) = (\alpha x_1, \alpha y_1)$$
for all $\alpha \in \mathbb{R}$ and $(x_1, x_2) \in \mathbb{R}^2$. Is $\mathbb{R}^2$ equipped with these operations a vector space? Justify your answer.

3. Decide whether $\{(s, 5s),\ s \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. How about $\{(s, s^2),\ s \in \mathbb{R}\}$?

4. Let $u = (4, 3, 2)$, $v = (1, 3, 5)$, $w = (3, 6, 9)$. Are u, v, w linearly dependent or linearly independent?

5. Let $u_1 = (-1, 1, 0, 2)$, $u_2 = (2, 0, 1, 1)$, $u_3 = (0, 2, 1, 5)$ and $u_4 = (0, 0, 2, 1)$. Are these vectors linearly dependent or independent?

6. Determine, whether a vector $z = (3, 2, 1)$ belongs to $\operatorname{span}\{u, v\}$ where $u = (1, 1, -1)$ and $v = (1, 2, 1)$.

7. Can $P(x) = x^3 + 2x + 1$ be expressed as a linear combination of $Q(x) = x^2 + x$ and $H(x) = x^3 - 2x^2 + 1$?

8. Can $f(x) = \tan x$ be expressed as a linear combination of $g(x) = \sin x$ and $h(x) = \frac{1}{\cos x}$?

9. Find $\alpha \in \mathbb{R}$ such that the vector $z = (1, 2, 0)$ is a linear combination of $u = (\alpha, -1, 1)$ and $v = (0, 2, 2)$.

10. Determine a dimension of $\operatorname{span}\{u, v, w\}$ where $u = (3, 0, 2)$, $v = (-1, 1, 0)$ and $w = (0, 3, 2)$.

11. Is $(-1, 5, 3) \in \operatorname{span}\{(1, 2, 2), (4, 1, 3)\}$. If yes, determine the coordinates with respect to the given basis.

12. Find coordinates of $x^2 + 2$ with respect to a basis $x^2 + 1$, $x - 2$, $2x^2 + x - 1$.

13. Find all α such that $u = (\alpha, 1, 0)$, $v = (1, \alpha, 1)$ and $w = (0, 1, \alpha)$ are linearly dependent.

1.2 Matrices – intro

1. Let $A = (1\ \ -2)$ and $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Compute AB and BA.

2. Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ and $B = (2\ \ 2\ \ -1)$. Compute AB^T.

3. Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & 0 & 5 \end{pmatrix}$. Compute AB and BA (if they exist).

4. Compute X
$$X = 3\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

5. Use elementary transformations to transform
$$A = \begin{pmatrix} 3 & 1 & 1 \\ 5 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix}$$
into an echelon form.

6. Find $\operatorname{rank} A$ where
$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 \\ 2 & 4 & 2 & -2 \end{pmatrix}.$$

7. Decide, whether vectors
$$\begin{aligned} u &= (2, 1, 1, 2) \\ v &= (3, 4, 4, 0) \\ w &= (0, 1, 0, -2) \\ z &= (-1, -1, 2, 2) \end{aligned}$$
are linearly dependent or independent.

8. Find all $\alpha \in \mathbb{R}$ such that
$$\operatorname{rank}\begin{pmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{pmatrix} = 2$$

1.3 Matrices – The Gauss elimination

1. Solve
$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

2. Solve
$$\begin{aligned} z + 3x &= y + 6 \\ x &= y + z \\ 2x - 3y &= 7 - z \end{aligned}$$

3. Solve
$$\begin{aligned} px + y - z &= 0 \\ x + (p-1)y + z &= 3 \\ x + 2y + z &= p \end{aligned}$$
for all real parameters p.

4. Find all solutions to
$$\begin{aligned} x + 2y - 3z + 2t &= 4 \\ 2x - y - z - t &= -2 \\ 5x - 5z &= 0 \\ -5y + 5z - 5t &= -5 \end{aligned}$$

5. Find A^{-1} for
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then use it to find solution to

$$\begin{aligned} 2x+y+z&=3\\ x+3z&=-7\\ 2x+y&=1 \end{aligned}$$

and a solution to

$$\begin{aligned} 2x+y+z&=0\\ x+3z&=3\\ 2x+y&=-1 \end{aligned}$$

1.4 Square matrices – intro

1. Find a matrix X fulfilling

$$X\begin{pmatrix}1&-1\\1&1\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}-1&1\end{pmatrix}=2\begin{pmatrix}3&2\\0&0\end{pmatrix}$$

2. Find all matrices X such that

$$\begin{pmatrix}2&3\\4&6\end{pmatrix}X+\begin{pmatrix}1&2\\-1&3\end{pmatrix}=\begin{pmatrix}7&8\\11&15\end{pmatrix}$$

1.5 Square matrices – determinants

1. Compute

$$\det\begin{pmatrix}2&-1&1\\1&3&2\\2&-2&-2\end{pmatrix}$$

2. Find all $\alpha\in\mathbb{R}$ for which the matrix

$$\begin{pmatrix}\alpha&3\\1&\alpha\end{pmatrix}$$

is singular.

3. Use the definition of determinant to compute

$$\det\begin{pmatrix}1&0&-2&1\\0&0&2&3\\0&1&0&5\\2&1&4&4\end{pmatrix}$$

4. Find the determinant of the matrix

$$\begin{pmatrix}1&2&5&7&10\\1&2&3&6&7\\1&1&3&5&5\\1&1&2&4&5\\1&1&1&1&1\end{pmatrix}$$

5. Use the Cramer rule to solve

$$\begin{aligned} x+2y-z&=0\\ -x+y+2z&=1\\ 2x+y-z&=1. \end{aligned}$$

6. Compute

$$\det\begin{pmatrix}2&0&0&-3\\0&1&-1&-1\\0&2&10&1\\-1&0&1&1\end{pmatrix}.$$

1.6 Eigenvalues

1. Write down the characteristic polynomial of

$$A=\begin{pmatrix}0&5&3\\-1&2&-1\\1&-5&-2\end{pmatrix}$$

and verify that $\lambda_1=-3$, $\lambda_2=2$, and $\lambda_3=4$ are its roots. Then compute the eigenvector corresponding to λ_2

2. Compute all eigenvalues of

$$A=\begin{pmatrix}1&-1&4\\3&2&-1\\2&1&-1\end{pmatrix}$$

and find the corresponding eigenvectors.

3. Find all eigenvalues of

$$A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

and find the corresponding eigenvectors including the generalized one.

4. Find all eigenvalues and eigenvectors of

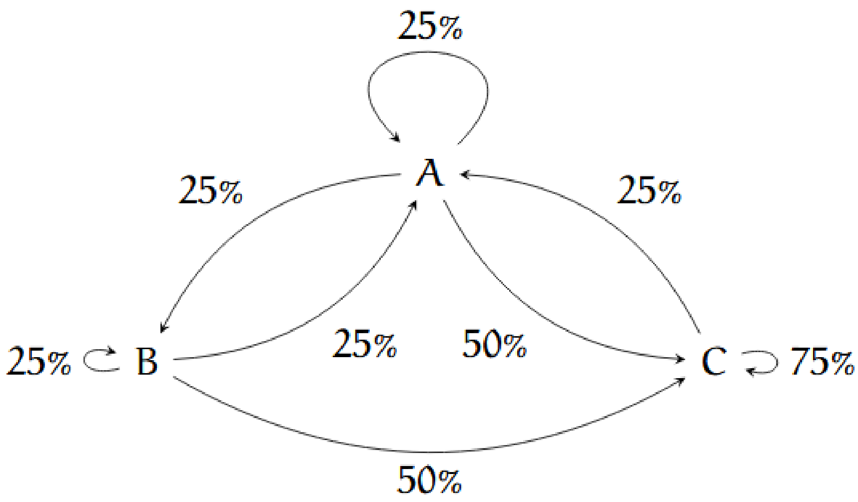
$$A=\begin{pmatrix}4&0&0\\-1&1&-5\\2&1&3\end{pmatrix}$$

5. A rental car company has three basis in three different cities A, B, C. The customers can pick up and return cars at any station. For example, a customer could pick up his car at A and return it in C.
We assume that there is only one car type available and that all customers rent their cars for one fixed time period (say one day). The graph below shows the proportions of cars that remain at a station or are transferred to other station over a day.

   (a) Find an appropriate matrix $P\in\mathbb{R}^3$ that describes this transition process, i.e., a matrix P such that $P\nu_0$ gives the distribution of cars after one day assuming ν_0 is the initial distribution.

   (b) How should the cars be distributed initially such that the distribution does not change over time? (i.e., find a distribution ν such that $P\nu=\nu$). Is it possible?

1.7 Square matrices – definiteness



1. Write the quadratic form Q whose matrix is

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 2 \\ -2 & 2 & 5 \end{pmatrix}$$

2. Write the matrix corresponding to the quadratic form

$$Q(x,y,z,t) = 2x^2+3y^2-xy-zt+2xt+5yz-4xz-6yt+z^2$$

3. Decide about the definiteness of the following forms

$$Q(x,y,z) = x^2+2xy+y^2-z^2+2(xz+yz)$$
$$Q(x,z,z) = 2x^2+2xy+2y^2+2xz+2z^2$$

4. Decide about the definiteness of the following matrices

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 0 \\ 3 & 5 & 6 \\ 0 & 6 & -4 \end{pmatrix}$$

1.8 Revision

1. Find all vectors v belonging to

   $\text{span}\{(1,-1,2),(1,2,0)\} \cap \text{span}\{(0,-1,-1),(-1,2,1)\}$.

2. Find coordinates of $(1,-1,1)$ with respect to a basis $(-1,1,0)$, $(0,-1,1)$ and $(1,0,-1)$.

3. Are vectors $(2,1,3)$, $(-1,2,1)$, $(0,-1,-1)$ linearly independent?

4. Decide about the rank of

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 5 & 7 & -4 \\ 1 & 1 & -2 \\ 3 & 4 & -3 \end{pmatrix}$$

5. What is the determinant of a square matrix where the first and fourth rows are the same?

6. The eigenvalues of

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

   are 2 and 3. Find the eigenvector and the generalized eigenvector corresponding to $\lambda = 2$.

7. Let there is a system of a thousand equations and thousand unknowns whose matrix is regular. Assume that the vector of the right hand side is the same as the column vector corresponding to the x_{485} variable. What is the solution to that system? Justify your claim.

8. Is there any solution (x,y,z) to an equation

$$x^2+2xy+2yz-3zx+2z^2+x^2?$$

2 Sequences and series

2.1 Sequences – intro

1. Find an explicit formula for a sequence given as

$$a_1 = 1, \quad a_{n+1} = (n+1)a_n.$$

2. Find an explicit formula for a sequence given as

$$a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{n}{n+1}a_n$$

   and verify your claim.

3. What is the explicit formula of

$$a_1 = \frac{1}{2}, \; a_{n+1} = a_n\frac{n}{n+2}?$$

   Verify your claim.

4. Try to write at least two recursive (implicit) formulas for $a_n = \frac{n+3}{n}$.

5. Decide about the monotonicity and boundedness of
$$a_n = \frac{n}{(n^2+3)}.$$
and verify your claim.
6. Find a sequence which is not bounded from below and also not bounded from above. Is the sequence monotone?
7. Decide about the monotonicity and boundedness of
$$a_n = \frac{n+1}{n^2}.$$
and verify your claim.
8. Decide about the monotonicity and boundedness of
$$a_n = \frac{n}{n^2+3}.$$
9. Decide about the monotonicity and boundedness of
$$a_n = \sqrt{n+1} - \sqrt{n}.$$

2.2 Sequences – limits

1. Solve
$$\lim \frac{n^2+1}{(1-n)(n+2)}$$
2. Solve
$$\lim \frac{(n^2+2)^{3/2}}{n-1}$$
3. Solve
$$\lim 2 - \frac{2^n+1}{3^n}$$
4. Solve
$$\lim \frac{1-n^3}{n+3}$$
5. Solve
$$\lim \frac{\sqrt{n^2+4n}-n}{5}$$
6. Solve
$$\lim \frac{\sqrt{n^2+2n+2}-n}{\sqrt{n}}$$
7. Solve
$$\lim \frac{2+n+3^n}{2^n+n^2-1}$$
8. Solve
$$\lim \left(\frac{n^2-4}{n^2+5}\right)^{n^2}$$
9. Solve
$$\lim \frac{\sqrt{n+2}\sqrt{2n+5}}{4n-3}$$
10. Solve
$$\lim \frac{(n+4)^8-(n^2+1)^4}{(2n-4)^7}$$
11. Solve
$$\lim \frac{n+(-1)^n}{2n+3}$$
12. Solve
$$\lim \frac{\left(\frac{1}{2}\right)^{n^2} - \left(\frac{1}{3}\right)^{n^2}}{\left(\frac{1}{2}\right)^{n^2+1} - \left(\frac{1}{3}\right)^{n^2+1}}$$
13. Solve
$$\lim (-1)^n \sqrt{n}(\sqrt{n+2} - \sqrt{n+1})$$
14. Solve
$$\lim \left(\frac{n}{n-2}\right)^n$$
15. Solve
$$\lim \sqrt{n^2+2n+2} - n$$
16. Solve
$$\lim \frac{(n+1)^4}{(n+\sqrt{n})^3}$$

2.3 Series – intro

1. Evaluate
$$\sum_{n=4}^{\infty} \frac{1}{3^n}.$$
2. Given that
$$\sum_{n=0}^{\infty} \frac{1}{n^3+1} = 1.6865,$$
evaluate
$$\sum_{n=2}^{\infty} \frac{1}{n^3+1}.$$
3. The partial sums are given as
$$s_n = \frac{n^2}{5+2n}.$$
Is the series convergent or divergent?
4. Find the value of
$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}}$$
5. Compute the first three partial sums of
$$\sum_{n=1}^{\infty} n2^n$$
and try to decide, whether is the sum convergent or divergent.
6. *Evaluate
$$\sum_{n=4}^{\infty} \frac{10}{n^2-4n+3}$$

2.4 Series – positive series

1. Examine the convergence of
$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}.$$
2. Decide about the convergence of
$$\sum_{n=4}^{\infty} \frac{n^2}{n^2-3}.$$
3. Examine the convergence of
$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2+2n}.$$
4. Examine
$$\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}.$$
(Hint: $\lim \sqrt[n]{n} = 1$.)
5. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9}.$$
6. Examine
$$\sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdots (1+3n)}{2 \cdot 6 \cdots (4n-2)}.$$
7. Examine the convergence of the following series
$$\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}.$$
8. Decide about the convergence of
$$\sum_{n=3}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$$
9. Examine the convergence of
$$\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}.$$
10. Examine
$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}.$$
11. Examine the convergence of
$$\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}.$$

2.5 Series – general sign

1. Decide about the convergence of
$$\sum_{n=1}^{\infty} \cos(\pi n) \frac{1}{n+3}.$$
2. Examine
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n+2}.$$
3. Examine the convergence of
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(n+1)^3-n^3}.$$
4. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+100}.$$
5. Examine
$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1}\right)^n.$$

3 Functions of multiple variables

3.1 Intro

1. Determine and sketch the domain of a function $f = x + \sqrt{y}$.
2. Determine and sketch the domain of a function $f = \sqrt[3]{\frac{1}{xy}}$.
3. Let $f(x, y) = 4x^2 + y$. Write a function
$$g(t) = f(2+t, 3+2t)$$
and draw its graph.
4. Determine and sketch the domain of a function $f(x, y) = \sqrt{|x| + |y| - 2}$.
5. Find and sketch the domain of a function
$$f(x, y) = \sqrt{16 - 4x^2 - y^2}.$$
What is the range of this function?
6. Determine and sketch the domain of a function $f(x, y) = \log(x \log(y - x))$.
7. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x, y) = xy$.
8. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x, y) = |x| + y$.
9. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x, y) = (x + y)^2$.
10. Find $g(t) = f(t, t+1)$ where $f(x, y) = x^2 + 2xy$. Draw the graph of $g(t)$.
11. Find $g(t) = f(t, t^2)$ where $f(x, y) = x - y^2$. Draw the graph of $g(t)$.
12. Show that $B_1(0, 0)$ is an open set.

13. Find $\partial\Omega$ for
$$\Omega = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2>1,\ |y| \le x\}.$$
14. Find a set whose boundary is
$$\{(x,y)\in\mathbb{R}^2, x^2+y^2=1\}\cup \{(x,y)\in\mathbb{R}^2, |x|=1,\ |y|\le 1\}\cup \{(x,y)\in\mathbb{R}^2,\ |y|=1,\ |x|\le 1\}$$
15. Determine, whether a set
$$M = \{(x,y)\in\mathbb{R}^2, |x|\ge -2, |y|<1,\ x^2<3\}$$
is open, closed or none of this. Justify your answer.

3.2 Limits and continuity

1. Compute $\lim_{(x,y)\to(0,0)} \frac{x+y}{x+y+1}$.
2. Compute $\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$.
3. Compute $\lim_{x\to\infty}\left(\lim_{y\to\infty} \frac{x+y}{x^2+y}\right)$ and $\lim_{y\to\infty}\left(\lim_{x\to\infty} \frac{x+y}{x^2+y}\right)$.
4. Determine, whether a function
$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} \text{ for } (x,y)\neq(0,0) \\ 0 \text{ for } (x,y)=(0,0). \end{cases}$$
is continuous.
5. * Examine a limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.
6. Examine $\lim_{(x,y)\to(0,0)} \frac{2xy}{xy+2x-y}$.
7. Compute $\lim_{(x,y)\to(1,2)} \frac{x^3y-xy^3+1}{(x-y)^2}$.
8. Examine $\lim_{(x,y)\to(2,2)} \frac{x^3-y^3}{x^4-y^4}$.
9. Compute $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$.
10. Compute $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$.

3.3 Derivatives – intro

1. Compute the derivative of $f(x,y) = \frac{x^2+1}{y^2+1}$ with respect to direction $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in a point $(1,0)$.
2. Compute $Df((2,1),(\cos\alpha,\sin\alpha))$ where
$$f(x,y) = x+xy$$
and $\alpha\in[0,2\pi)$ is a parameter.
3. Compute ∇x^y.
4. Compute
$$\nabla x\sin(x+y).$$
5. Compute all first order partial derivatives of
$$f(x,y,z) = e^y x^2 + \frac{y}{z}$$
6. Compute all partial derivatives of the first and second order of a function
$$f = xy + \frac{x}{y}.$$
7. Compute all second derivatives of
$$f(x,y) = \arctan\frac{x+y}{1-xy}.$$
8. Compute the first and the second order partial derivatives of a function
$$f(x,y) = \frac{y}{\sin x}.$$
9. Determine all first order derivatives of
$$f(x,y) = (3x+2y)^{\log y}.$$
10. Determine the tangent plane to
$$f(x,y) = x^2-2xy-3y^2$$
at point $A=(-1,1)$.
11. Write the second degree Taylor polynomial at point $(1,3)$ of a function
$$f(x,y) = x^2y+y^2.$$
12. Use the second degree Taylor polynomial to determine an approximate value of
$$\sqrt{20-(1.9)^2-6.6}.$$

3.4 Implicitly given functions

1. Show that there is a function $y(x)$ determined by
$$x^2+y^2+xy=3$$
in the vicinity of $(1,1)$. Compute the first and the second derivative of $y(x)$ at $x=1$.
2. Show that there is a function $y(x)$ determined by
$$y-\frac{1}{2}siny = x$$
in the neighborhood of $\left(\frac{\pi-1}{2}, \frac{\pi}{2}\right)$. Decide, whether $y(x)$ is convex or concave in some neighborhood of the given point.
3. Consider an equation
$$-x^2+y^2-2xy+y=0.$$
Find points satisfying the implicit function theorem and identify those of them which are stationary points of $y(x)$. Decide, whether there are local extremes at the stationary points.

3.5 Extremes

1. Find local extremes of the following functions

$$\begin{aligned} a(x,y) &= x^2 - xy + y^2 - 2x + y, \\ b(x,y) &= e^{-x^2-y^2}, \\ c(x,y) &= (\sin x)^2 (y-1)^2, \\ d(x,y) &= e^{2x+3y}(8x^2 - 6xy + 3y^2), \\ e(x,y) &= x^4 + y^4 - x^2 - 2xy - y^2, \\ f(x,y) &= xy\sqrt{1-x^2-y^2}. \end{aligned}$$

2. Find the maximum and the minimum of

$$f(x,y) = \cos^2 x + \cos^2 y$$

on $M = \{(x,y) \in \mathbb{R}^2,\ x - y = \frac{\pi}{4}\}$.

3. Find the maximum and the minimum of

$$f(x,y) = x^2 y$$

on $M = \{(x,y) \in \mathbb{R}^2,\ x + 2y = 1\}$.

4. Find the maximum and the minimum of

$$f(x,y) = 4xy$$

subject to the constraint $M = \left\{(x,y) \in \mathbb{R}^2,\ \frac{x^2}{9} + \frac{y^2}{16} = 1\right\}$.

5. Find the maximum and the minimum of

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraint $M = \{(x,y,z) \in \mathbb{R}^3,\ x^4 + y^4 + z^4 = 1\}$

6. Find the maximum and the minimum of

$$f(x,y) = x^2 + 2y$$

on the triangle with vertices $A = (-1,-1)$, $B = (5,-1)$ and $C = (-1,5)$.

7. Find the maximum and the minimum of

$$f(x,y) = x^2 + xy + 2y$$

on $M = \{(x,y) \in \mathbb{R}^2,\ y \geq x^2,\ y \leq 9\}$.

8. Find the maximum and the minimum of

$$f(x,y,z) = 2x + 3y + 5z$$

on the ball $M = \{(x,y,z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 19\}$.

9. Find the maximum and the minimum of

$$f(x,y,z) = xyz$$

on $M = \{(x,y,z) \in \mathbb{R}^3,\ x \geq 0,\ x + 9y^2 + z^2 \leq 4\}$

10. Find the maximum and the minimum values of $f(x,y) = e^{xy}$ subject to the constraint

$$x^3 + y^3 - 16 = 0.$$

3.5.1 Applications

1. Production is modeled by the function $f = 75 l^{3/4} c^{1/4}$ where l is the units of labor and c is the units of capital. Each unit of labor costs \$ 100 and each unit of capital costs \$ 125. If a company has \$ 20 000 to spend, how many units of labor and capital should be purchased.
2. Find three positive number whose sum is 48 and whose product is as large as possible.
3. The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box?
4. What is the least amount of fence required to make a yard bordered on one side by a house? The required area of the yard is $72\,m^2$.

4 Systems of ODEs

4.1 Linear equations – homogeneous case

1. Write the given system as in the matrix form

$$\begin{aligned} x'(t) &= 3x(t) - y(t) + t^2 \\ y'(t) &= e^t - x(t) + 2y(t). \end{aligned}$$

2. Express the given system of higher order differential equations as a matrix system:

$$\begin{aligned} x'' + 3x' - y' + 2y &= 0 \\ y'' + x' + 3y' + y' &= 0 \end{aligned}$$

3. Rewrite the equation

$$x'''(t) - 2x''(t) + x'(t) + 2x(t) = t$$

into a system of first-order linear differential equations.

4. Find the fundamental solution set for a system

$$x'(t) = Ax$$

where

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

5. Find the fundamental solution set for

$$x'(t) = Ax(t), \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}.$$

6. Find the general solution to

$$x'(t) = Ax(t), \text{ where } A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}.$$

Then choose the particular solution satisfying $x(0) = (-1, 0)$.

7. Find the general solution to

$$x'(t) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} x(t).$$

8. Find the fundamental solution system

$$x'(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x(t).$$

9. Find the solutions to

$$x'(t) = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} x(t).$$

which satisfies

$$x(0) = (3, -1, 0).$$

10. Find a general solution to

$$x'(t) = \begin{pmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{pmatrix} x(t).$$

11. Find all solutions to

$$x'(t) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} x(t).$$

12. Find all solutions to

$$x'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} x(t).$$

4.2 Linear equations – nonzero right hand side

1. Solve

$$x'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} -t-1 \\ -4t-2 \end{pmatrix}.$$

2. **Conventional Combat Model**: A simplistic model of a pair of conventional forces in combat yields the following system:

$$x'(t) = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} x + \begin{pmatrix} p \\ q \end{pmatrix}$$

where x_1 and x_2 represent the strength of oposing forces at time t. The terms $-ax_1$ and $-dx_2$ represent the operational loss rates and the terms $-bx_2$ and $-cx_1$ represent the combat loss rates for the troops x_1 and x_2 respectively. The constants p and q represent the respective rates of reinforcements. Let $a = 1$, $b = 4$, $c = 3$, $d = 2$ and $p = q = 5$. By solving the appropriate initial value problem, determine which forces will win if

   (a) $x(0) = (20, 20)$.

   (b) $x(0) = (21, 20)$.

   (c) $x(0) = (20, 21)$.

3. Find the solution to

$$x'(t) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}.$$

satisfying $x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

4. Find the solution to

$$x' = \begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} x + \begin{pmatrix} te^{2t} \\ e^{2t} \end{pmatrix}$$

satisfying $x_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

5. Consider a typical home with attic, basement and insulated main floor. Denote

$$\begin{aligned} x(t) &- \text{ Temperature in the attic,} \\ y(t) &- \text{ Temperature in the main living area,} \\ z(t) &- \text{ Temperature in the basement,} \\ t &- \text{ Time in hours.} \end{aligned}$$

Assume it is winter time and the outside temperature is 2^0C and assume that the basement earth temperature is 7^0C (thus the initial values are $x(0) = y(0) = 2$ and $z(0) = 7$). A small electric heater is turned on at noon. It provides a 10^0C rise per hour. The temperatures are governed by

$$\begin{aligned} x' &= k_0(45 - x) + k_1(y - x) \\ y' &= k_1(x - y) + k_2(35 - y) + k_3(z - y) + 10 \\ z' &= k_3(y - z) + k_4(35 - z). \end{aligned}$$

The insulation constant will be defined as

$$k_0 = k_1 = k_4 = \frac{1}{2}, \quad k_2 = k_3 = \frac{1}{4}.$$

Compute the evolution of temperatures in the attic, in the main living area and in the basement.

6. Let R denote the affection that Romeo has for Juliet and J be the affection that Juliet has for Romeo. Positive values indicate love and negative values indicate dislike.

One possible model is given by

$$\begin{aligned} R' &= bJ \\ J' &= cR \end{aligned}$$

with $b > 0$ and $c < 0$. In this case Romeo loves Juliet the more she likes him. But Juliet backs away when she finds his love for her increasing.

A typical system relating the combined changes in affection can be modeled as

$$\begin{aligned} R' &= aR + bJ \\ J' &= cR + dJ \end{aligned}$$

Compute the evolution of their love from the initial value $(1,1)$ (the love they felt when they first met) and try to describe their destiny in case of

(a) $a = b = c = d = 1$, i.e., Romeo gets more excited by Juliet's love for him and Juliet feels it in same way.

(b) $a = b = d = 1$ and $c = -1$, i.e., Romeo feels it similarly as above, but Juliet backs as Romeo's love increases.

(c) $a = 0$, $b = 2$, $c = 0$ and $d = 1$, i.e., Romeo is falling in love to Juliet the more she likes him. This affects also the Juliet's passion but not in the same way.

4.3 Phase plane

1. In problems below, find the critical point set for the given system

   (a)
   $$x' = x - y$$
   $$y' = x^2 + y^2 - 1$$

   (b)
   $$x' = x^2 - 2xy$$
   $$y' = 3xy - y^2$$

2. Find all the critical points of the system
   $$x' = x^2 - 1$$
   $$y' = xy$$
   and sketch its trajectories in a phase plane.

3. Find all the critical points of the system
   $$x' = 2y$$
   $$y' = 2x$$
   and sketch its trajectories in a phase plane.

4. Find all the critical points of the system
   $$x' = (y - x)(y - 1)$$
   $$y' = (x - y)(x - 1).$$
   and sketch its trajectories in a phase plane.

5. For the systems below, classify the critical point at the origin.

   (a)
   $$x' = 3x + 5y$$
   $$y' = -5x - 5y$$

   (b)
   $$x' = 2x + 13y$$
   $$y' = -x - 2y$$

   (c)
   $$x' = 6x - y$$
   $$y' = 8y$$

6. Find and classify the critical points of the given linear system

   (a)
   $$x' = -4x + 2y + 8$$
   $$y' = x - 2y + 1$$

   (b)
   $$x' = 2x + y + 9$$
   $$y' = -5x - 2y - 22$$

   (c)
   $$x' = 2x + 4y + 4$$
   $$y' = 3x + 5y + 4$$

7. Show that the system
   $$x' = x - y$$
   $$y' = 3x + 5y - x^3$$
   is almost linear near origin.

8. Find all the critical points for
   $$x' = 16 - xy$$
   $$y' = x - y^3$$
   and discuss their type and stability.

5 Difference equations

1. Find all solutions to
   $$y(n+3) + y(n+2) - 8y(n+1) - 12y(n) = 0.$$

2. Find all solutions to
   $$y(n+2) - 5y(n+1) + 6y(n) = 1 + n.$$

3. Find all solutions to
   $$y(n+2) - 5y(n+1) + 4y(n) = 4^n - n^2.$$

4. Solve
   $$y_{n+3} + y_{n+2} - 8y_{n+1} - 12y_n = 0.$$

5. Find the solution to
   $$y_{n+2} - 5y_{n+1} + 6y_n = 1 + n$$
   satisfying $y(0) = 2$, $y(1) = 3$.

6. Find all solutions to
   $$y_{n+2} - 5y_{n+1} + 4y_n = 4^n - n^2.$$