1 Extremes

1.1 Reminder

1. Does a function $f(x) = x^3 + 2x^2 - x$ attain its global maximum or minimum? Justify your answer.

2. Find all local extrema of $f(x) = (x^2-4x+3)e^x$. Decide which of these are global extrema.

3. Find the maximum and the minimum of $f(x) = x^4 + 4x^3 - 14x^2$ attained on the set $[0, 3]$

4. Find and identify local extrema of $f(x, y) = x^3 - 3x - y^2 + 4y$.

5. Find and identify local extrema of $f(x, y, z) = x^3 - 2x^2 + y^2 + z^2 - 2xy + xz - yz + 3z$.

6. To reduce shipping distances between the manufacturing facilities and a major consumer, a Korean computer brand, Intel Corp. intends to start production of a new controlling chip for Pentium III microprocessors at their two Asian plants. The cost of producing x chips at Chiangmai (Thailand) is
$$C_1 = 0.002x^2 + 4x + 500$$
and the cost of producing y chips at Kuala-Lumpur (Malaysia) is
$$C_2 = 0.005x^2 + 4x + 275.$$
The Korean computer manufacturer buys them for \$150 per chip. Find the quantity that should be produced at each Asian location to maximize the profit if, in accordance with Intel's marketing department, it is described by the expression:
$$P(x, y) = 150(x + y) - C_1 - C_2.$$

7. The profit obtained by producing x units of A and y units of B is approximated by
$$P(x, y) = 8x + 10y - 0.001(x^2 + xy + y^2) - 10\,000.$$
Find the level that produces a maximum profit.

1.2 Constraints

1. Find the maximum and the minimum of
$$f(x, y) = x^2y$$
on $M = \{(x, y) \in \mathbb{R}^2,\ x + 2y = 1\}$.

2. Find the maximum and the minimum of
$$f(x, y) = \cos^2 x + \cos^2 y$$
on $M = \{(x, y) \in \mathbb{R}^2,\ x - y = \frac{\pi}{4}\}$.

3. Find the maximum and the minimum of
$$f(x, y) = x^2 + 2y$$
on the triangle with vertices $A = (-1, -1)$, $B = (5, -1)$ and $C = (-1, 5)$.

4. Determine a maximum and minimum of $f(x, y) = -y^2 + x^2 + \frac{4}{3}x^3$ on the set $M = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 4\}$.

5. Find the maximum and the minimum of
$$f(x, y) = 4xy$$
subject to the constraint $M = \left\{(x, y) \in \mathbb{R}^2,\ \frac{x^2}{9} + \frac{y^2}{16} = 1\right\}$.

6. Find the maximum and the minimum of
$$f(x, y, z) = x^2 + y^2 + z^2$$
subject to the constraint $M = \{(x, y, z) \in \mathbb{R}^3,\ x^4 + y^4 + z^4 = 1\}$

7. Find the maximum and the minimum of
$$f(x, y) = x^2 + xy + 2y$$
on $M = \{(x, y) \in \mathbb{R}^2,\ y \geq x^2,\ y \leq 9\}$.

8. Find the maximum and the minimum of
$$f(x, y, z) = 2x + 3y + 5z$$
on the ball $M = \{(x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 19\}$.

9. Find the maximum and the minimum of
$$f(x, y, z) = xyz$$
on $M = \{(x, y, z) \in \mathbb{R}^3,\ x \geq 0,\ x + 9y^2 + z^2 \leq 4\}$.

1.2.1 Applications

1. Production is modeled by the function $f = 75l^{3/4}c^{1/4}$ where l is the units of labor and c is the units of capital. Each unit of labor costs \$ 100 and each unit of capital costs \$ 125. If a company has \$ 20 000 to spend, how many units of labor and capital should be purchased.

2. Find three positive number whose sum is 48 and whose product is as large as possible.

3. A large container in the shape of a rectangular solid must have a volume of $480\,m^3$. The bottom of the container costs $\$5/m^2$ to construct whereas the top and sides cost $\$3m^2$ to construct. Find the dimensions of the container of this size has the minimum cost.

4. The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box?

5. What is the least amount of fence required to make a yard bordered on one side by a house? The required area of the yard is $72\,m^2$.

2 Integrals

2.1 Basic methods

1. Compute
$$\int \left(\frac{1-x}{x}\right)^2 \mathrm{d}x$$

2. Compute
$$\int (1+\sin x+\cos x)\,\mathrm{d}x$$

3. Compute
$$\int \left(\sqrt{x}+\frac{1}{x}\right)^2 \mathrm{d}x$$

4. Compute
$$\int \left(\frac{2}{x}+\frac{2}{x^2}+\frac{2}{x^3}\right) \mathrm{d}x$$

5. Compute
$$\int \frac{(1-x)^3}{x\sqrt[3]{x}}\,\mathrm{d}x$$

6. Compute
$$\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}\,\mathrm{d}x$$

7. Compute
$$\int (1-x)(1-2x)(1-3x)\,\mathrm{d}x$$

8. Compute
$$\int \left(2^x+3^x\right)^2\,\mathrm{d}x$$

9. Compute
$$\int \frac{x^2}{1-x^2}\,\mathrm{d}x$$

10. Compute
$$\int \frac{(\sqrt{2x}-\sqrt[3]{3x})^2}{x}\,\mathrm{d}x$$

2.2 Substitution

1. Compute
$$\int 3x\sqrt{5+x^2}\,\mathrm{d}x$$

2. Compute
$$\int x^3\sqrt[3]{1+x^2}\,\mathrm{d}x$$

3. Compute
$$\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\frac{1}{1+x}\,\mathrm{d}x$$

4. Compute
$$\int x^3(1-5x^2)^{10}\,\mathrm{d}x$$

5. Compute
$$\int \frac{\log x-2}{x\sqrt{\log x}}\,\mathrm{d}x$$

6. Compute
$$\int \frac{e^{2/x}}{x^2}\,\mathrm{d}x$$

2.3 Integration by parts

1. Compute
$$\int xe^{-x}\,\mathrm{d}x$$

2. Compute
$$\int x^2\sin 2x\,\mathrm{d}x$$

3. Compute
$$\int \arcsin x\,\mathrm{d}x$$

4. Compute
$$\int x^2e^{-2x}\,\mathrm{d}x$$

5. Compute
$$\int \log x\,\mathrm{d}x$$

6. Compute
$$\int \sqrt{x}\log^2 x\,\mathrm{d}x$$

2.4 Both methods at once

1. Compute
$$\int \left(\frac{\log x}{x}\right)^2 \mathrm{d}x$$

2. Compute
$$\int e^{\sqrt{x}}\,\mathrm{d}x$$

3. Compute
$$\int x\sin^2 x\,\mathrm{d}x$$

4. Compute
$$\int x(\arctan^2 x)\,\mathrm{d}x$$

2.5 Rational functions

1. Compute
$$\int \frac{2x+3}{(x-2)(x+5)}\,\mathrm{d}x$$
2. Compute
$$\int \frac{x^{10}\,\mathrm{d}x}{x^2+x-2}$$
3. Compute
$$\int \frac{\mathrm{d}x}{(x+1)(x^2+1)}$$
4. Compute
$$\int \frac{x^2+1}{(x+1)^2(x-1)}\,\mathrm{d}x$$
5. Compute
$$\int \frac{x}{x^3-1}\,\mathrm{d}x$$
6. Compute
$$\int \frac{\mathrm{d}x}{x^5-x^4+x^3-x^2+x-1}$$
7. Compute
$$\int \frac{x^3}{x^2+4}\,\mathrm{d}x$$
8. Compute
$$\int \frac{6x^3+6}{x^3-5x^2+6x}\,\mathrm{d}x$$
9. Compute
$$\int \frac{4x+5}{x^2+4x+5}\,\mathrm{d}x$$
10. Compute
$$\int \frac{x^3+2x^2}{x^2+2x+1}\,\mathrm{d}x$$

2.6 Newton's and Riemann's integral

1. Find the area of the region enclosed by the following curves: $y=e^x$, $y=x^2-1$, $x=1$, and $x=-1$.
2. Find the area of the region inbetween of $y=e^x$, $y=xe^x$, $x=0$.
3. Find the area of $M=\{(x,y)\in\mathbb{R}^2,\, y\le \log x\, y\ge \log^2 x\}$.
4. Find the area of the region bounded by $y^2=2x+1$, $y=x-1$.
5. Compute
$$\int_{-1}^{5} \frac{1}{3x+5}\,\mathrm{d}x.$$
6. Compute
$$\int_0^2 x\sqrt{4-x^2}\,\mathrm{d}x.$$
7. Find the area of the region bounded by $y=2x^3$ and $y=\frac{x}{2}$.
8. Compute
$$\int_0^2 xe^{1-x^2}\,\mathrm{d}x.$$
9. Find the area of the region bounded by $y=\frac{1}{1+x^2}$ and $y=\frac{x^2}{2}$
10. Find the area of the region bounded by $y=|\log x|$, $y=0$, $x=\frac{1}{10}$ and $x=10$.

2.7 Probability

1. Let the random variable X have the density $f(x)=\frac{x^2}{9}\chi_{[0,3]}$. Find $E(X)$ (expectation of the value of the random variable).
2. The waiting time for a beer at the U Lva restaurant is on average 5 minutes. Determine the probability that we will wait for a beer for more than 12 minutes. What is the waiting time during which a customer will be served with a probability of 0.9? (Assume the waiting time is exponentially distributed.)
3. Scores on an intelligence test for the age group 20 to 34 are approximately normally distributed with mean 110 and standard deviation 25. About what percent of people in this group have scores above 160? If only 1% of people in this age group have IQs higher than Elizabeth, what is Elizabeth IQ?
4. The Edwards's Theater chain has studied its movie customers to determine how much money they spend on concessions. The study revealed that the spending distribution is approximately normally distributed with a mean of \$4.11 and a standard deviation of \$1.37. What percentage of customers will spend less than \$3.00 on concessions?

2.8 Double integrals

1. Write M_x and M_y for
$$M=\{(x,y)\in\mathbb{R}^2,\, x^2+y^2\le 16,\, x\ge y\}.$$
2. Change the order of integration of
$$\int_0^2\left(\int_x^{2x} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$
3. Compute
$$\int_M 1\,\mathrm{d}x\mathrm{d}y,$$
where $M=\{(x,y)\in\mathbb{R}^2,\, x^2\le y\le 4-x^2\}$.
4. Compute
$$\int_M x^2+y\,\mathrm{d}x\mathrm{d}y,$$
where M is a triangle with vertices $(0,0)$, $(1,3)$, and

(3, 1).

5. Compute
$$\int_M xy^2 \,\mathrm{d}x\mathrm{d}y,$$
where $M = \{(x,y) \in \mathbb{R}^2,\ x^2 \le y \le x\}$.

6. Evaluate
$$\int_M 5x^3 \cos(y^3) \,\mathrm{d}x\mathrm{d}y,$$
where M is the region bounded by $y = 2$ and $y = \frac{1}{4}x^2$.

7. Evaluate
$$\int_M \frac{1}{\sqrt[3]{y}(x^3+1)} \,\mathrm{d}x\mathrm{d}y,$$
where M is the region bounded by $x = -y^{\frac{1}{3}}$, $x = 3$, and x-axis.

8. Compute the Jacobian determinant of
$$\Phi(u,v) = (u^2v^3, 4 - 2\sqrt{u})$$

9. Evaluate the integral
$$\int_M 3x - 2y \,\mathrm{d}x\mathrm{d}y$$
over the parallelogram M bounded by lines $y = \frac{3}{2}x - 4$, $y = \frac{3}{2}x + 2$, $y = -2x + 1$, and $y = -2x + 3$.

10. Evaluate
$$\int_M xy^3 \,\mathrm{d}x\mathrm{d}y$$
where M is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ and $y = 6$ using $x = \frac{v}{6u}$, $y = 2u$.

11. Compute
$$\int_M x + 2y \,\mathrm{d}x\mathrm{d}u$$
where M is the triangle with vertices $(0,3)$, $(4,1)$, and $(2,6)$ using the transformation $x = \frac{1}{2}(u - v)$, $y = \frac{1}{4}(3u + v + 12)$.

12. Evaluate the integral
$$\int_M \sin(x^2 + y^2) \,\mathrm{d}x\mathrm{d}y$$
where M is a disk with radius 2.

13. Compute
$$\int_M e^{-x^2-y^2} \,\mathrm{d}x\mathrm{d}y$$
where $M = \{(x,y) \in \mathbb{R}^2,\ x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 100\}$.

14. Evaluate
$$\int_M y^2 + 3x \,\mathrm{d}x\mathrm{d}y$$
where M is the region in the 3rd quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

15. Compute
$$\int_M \sqrt{1 + 4x^2 + 4y^2} \,\mathrm{d}x\mathrm{d}y$$
where M is the bottom half of $\{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 \le 16\}$

3 Sequences and series

3.1 Intro

1. Find the explicit formula for the sequence given as
$$a_1 = 1,\ a_{n+1} = (n+1)a_n.$$

2. Find the explicit formula for the sequence
$$a_1 = \frac{1}{2},\ a_{n+1} = \frac{n}{n+1}a_n.$$

3. What is the explicit formula of
$$a_1 = \frac{1}{2},\ a_{n+1} = a_n\frac{n}{n+2}?$$
Verify your claim.

4. Decide about the monotonicity of
$$a_n = \frac{n}{n^2+3}.$$

5. Decide about the monotonicity of
$$a_n = \frac{n+1}{n^2}.$$

6. Decide about the monotonicity and boundedness of
$$a_n = \sqrt{n+1} - \sqrt{n}.$$
and verify your claim.

7. Find a sequence which is not bounded from below and also not bounded from above. Is the sequence monotone?

3.2 Limits

1. $$\lim \frac{\sqrt{n^2+2n+2} - n}{\sqrt{n}}$$

2. $$\lim \left(\frac{n}{n-2}\right)^n$$

3. $$\lim (-1)^n \sqrt{n} \left(\sqrt{n+2} - \sqrt{n}\right)$$

4. $$\lim \left(\frac{2n-1}{2n+1}\right)^{n^2}$$

5. $$\lim 2 - \frac{2^n+1}{3^n}$$

6.
$$\lim \frac{n+(-1)^n}{2n+3}$$

7. Solve
$$\lim \sqrt{n^2+2n+2}-n$$

8. Solve
$$\lim \frac{(n+1)^4}{(n+\sqrt{n})^3}$$

3.3 Series – intro

1. Evaluate
$$\sum_{n=4}^{\infty} \frac{1}{3^n}.$$

2. Given that
$$\sum_{n=0}^{\infty} \frac{1}{n^3+1} = 1.6865,$$
evaluate
$$\sum_{n=2}^{\infty} \frac{1}{n^3+1}.$$

3. The partial sums are given as
$$s_n = \frac{n^2}{5+2n}.$$
Is the series convergent or divergent?

4. Find the value of
$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}}$$

5. Compute the first three partial sums of
$$\sum_{n=1}^{\infty} n2^n$$
and try to decide, whether is the sum convergent or divergent.

6. *Evaluate
$$\sum_{n=4}^{\infty} \frac{10}{n^2-4n+3}$$

3.4 Series – positive series

1. Examine the convergence of
$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}.$$

2. Decide about the convergence of
$$\sum_{n=4}^{\infty} \frac{n^2}{n^2-3}.$$

3. Examine the convergence of
$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2+2n}.$$

4. Examine
$$\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}.$$
(Hint: $\lim \sqrt[n]{n} = 1$.)

5. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9}.$$

6. Examine
$$\sum_{n=1}^{\infty} \frac{4\cdot 7\cdots(1+3n)}{2\cdot 6\cdots(4n-2)}.$$

7. Examine the convergence of the following series
$$\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}.$$

8. Decide about the convergence of
$$\sum_{n=3}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$$

9. Examine the convergence of
$$\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}.$$

10. Examine
$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}.$$

11. Examine the convergence of
$$\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}.$$

3.5 Series – general sign

1. Decide about the convergence of
$$\sum_{n=1}^{\infty} \cos(\pi n)\frac{1}{n+3}.$$

2. Examine
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n+2}.$$

3. Examine the convergence of
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(n+1)^3-n^3}.$$

4. Decide about the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+100}.$$

5. Examine

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1} \right)^n.$$