1 Introduction

1. Complete the following table:

A	B	C	$A \vee (\neg C)$	$(A \& B) \vee C$	$A \Rightarrow (B \Rightarrow C)$	$A \vee (B \Leftrightarrow C)$
true	true	true				
true	true	false				
true	false	true				
true	false	false				
false	true	true				
false	true	false				
false	false	true				
false	false	false				

2. Have three propositions: $A =$'Kutná Hora is the capital of Czechia', $B =$'Praha is the capital of Czechia', $C =$'two plus two is four' and $D =$'Pigs can fly'. Write down the following sentences and decide about their validity:

   (a) $A \vee B$.
   (b) $A \Leftrightarrow D$.
   (c) $A \Rightarrow C$.
   (d) $C \Rightarrow A$.
   (e) $B \vee D$.
   (f) $B \& C$.
   (g) $\neg A \& C$.

3. Quantifiers:
   We define the following

   - a: Anastazia
   - b: Bart
   - c: Cicero
   - $B(x, y)$: x belongs to y
   - $D(x, y)$: x hates y
   - $C(x)$: x is a cat
   - $F(x)$: x is wild
   - $P(x)$: x is a human.

   Try to rewrite the following formulas into sentences (try to make the sentences as nice as possible):

   (a) $C(b) \& F(b) \& B(b, c)$
   (b) $\forall x,\ (C(x) \Rightarrow D(a, x))$
   (c) $\exists x,\ (C(x) \& F(x) \& B(x, y))$
   (d) $\forall x, \forall y,\ ((C(x) \& F(x)) \Rightarrow (P(y) \Rightarrow D(y, x)))$
   (e) $\forall x,\ (C(x) \Rightarrow \exists y,\ (P(y) \& B(x, y)))$
   (f) $\neg \exists x,\ (C(x) \& B(x, a)) \& \exists x,\ (F(x) \Rightarrow D(a, x))$.

4. Sets

   (a) Find $\sup A$ and $\inf A$ for $A = \left\{ \frac{p}{p+q},\ p, q \in \mathbb{N} \right\}$.
   (b) Show that $\sup[0, 2] = \sup(0, 2) = 2$.
   (c) Let $A, B \subset \mathbb{R}$ be nonempty sets. Try to express $\sup(A \cup B)$ and $\sup(A \cap B)$ by $\sup A$ and $\sup B$, if it is possible.

5. Complex numbers: compute

   (a) i^3, i^{221}, $-i^{-3}$, i^{-5}, $(-i)^4$
   (b) $(1 + 5i)(2 - i)$, $(1 - i)(1 + i)$
   (c) $\frac{3-i}{4+i}$
   (d) $|5 + 12i| \frac{5-12i}{3-2i}$

6. Math induction

   (a) Prove that for all $n \in \mathbb{N}$ it holds that
   $$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$
   (b) Prove that six is a divisor of $n^3 + 5n$ for every $n \in \mathbb{N}$.
   (c) Prove that
   $$(1 + x)^n \geq 1 + nx$$
   for every $x > -1$ and every $n \in \mathbb{N}$.

2 Linear algebra

2.1 Vector spaces

1. Let $u = (1, 2, 3)$, $v = (-3, 1, -2)$ and $w = (2, -3, -1)$. Compute $u + v$, $u + v + w$, $2u + 2v + w$.

2. We define operations $\oplus$ and $\odot$ on $\mathbb{R}^2$ such that
$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, 0),\ \alpha \odot (x_1, y_1) = (\alpha x_1, \alpha y_1)$$
for all $\alpha \in \mathbb{R}$ and $(x_1, x_2) \in \mathbb{R}^2$. Is $\mathbb{R}^2$ equipped with these operations a vector space? Justify your answer.

3. Decide whether $\{(s, 5s),\ s \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. How about $\{(s, s^2),\ s \in \mathbb{R}\}$?

4. Let $u = (4, 3, 2)$, $v = (1, 3, 5)$, $w = (3, 6, 9)$. Are u, v, w linearly dependent or linearly independent?

5. Let $u_1 = (-1, 1, 0, 2)$, $u_2 = (2, 0, 1, 1)$, $u_3 = (0, 2, 1, 5)$ and $u_4 = (0, 0, 2, 1)$. Are these vectors linearly dependent or independent?

6. Determine, whether a vector $z = (3, 2, 1)$ belongs to $\text{span}\{u, v\}$ where $u = (1, 1, -1)$ and $v = (1, 2, 1)$.

7. Can $P(x) = x^3 + 2x + 1$ be expressed as a linear combination of $Q(x) = x^2 + x$ and $H(x) = x^3 - 2x^2 + 1$?

8. Can $f(x) = \tan x$ be expressed as a linear combination of $g(x) = \sin x$ and $h(x) = \frac{1}{\cos x}$?

9. Find $\alpha \in \mathbb{R}$ such that the vector $z = (1, 2, 0)$ is a linear combination of $u = (\alpha, -1, 1)$ and $v = (0, 2, 2)$.

10. Determine a dimension of $\text{span}\{u, v, w\}$ where $u = (3, 0, 2)$, $v = (-1, 1, 0)$ and $w = (0, 3, 2)$.

11. Is $(-1, 5, 3) \in \text{span}\{(1, 2, 2), (4, 1, 3)\}$. If yes, determine the coordinates with respect to the given basis.

12. Find coordinates of $x^2 + 2$ with respect to a basis $x^2 + 1$, $x - 2$, $2x^2 + x - 1$.

13. Find all α such that $u = (\alpha, 1, 0)$, $v = (1, \alpha, 1)$ and $w = (0, 1, \alpha)$ are linearly dependent.

2.2 Matrices – intro

1. Let $A = (1\ \ -2)$ and $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Compute AB and BA.

2. Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ and $B = (2\ \ 2\ \ -1)$. Compute AB^T.

3. Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & 0 & 5 \end{pmatrix}$. Compute AB and BA (if they exist).

4. Compute X
$$X = 3\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

5. Use elementary transformations to transform
$$A = \begin{pmatrix} 3 & 1 & 1 \\ 5 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix}$$
into an echelon form.

6. Find $\text{rank}A$ where
$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 \\ 2 & 4 & 2 & -2 \end{pmatrix}.$$

7. Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -2 \end{pmatrix}$. Compute
$$A + 2B,\ AB,\ A + \begin{pmatrix} 1 \\ -1 \end{pmatrix} C, B - 3C^T \begin{pmatrix} 1 & -1 \end{pmatrix},\ A - CB.$$

8. Decide, whether vectors
$$\begin{aligned} u &= (2, 1, 1, 2) \\ v &= (3, 4, 4, 0) \\ w &= (0, 1, 0, -2) \\ z &= (-1, -1, 2, 2) \end{aligned}$$
are linearly dependent or independent.

9. Find all $\alpha \in \mathbb{R}$ such that
$$\text{rank}\begin{pmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{pmatrix} = 2$$

10. Decide, whether $u = (1, 1, -1)$, $v = (2, 1, 0)$ and $w = (-1, 3, 1)$ span $\mathbb{R}^3$.

2.3 Matrices – The Gauss elimination

1. Solve
$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

2. Solve
$$\begin{aligned} z + 3x &= y + 6 \\ x &= y + z \\ 2x - 3y &= 7 - z \end{aligned}$$

3. Solve
$$\begin{aligned} px + y - z &= 0 \\ x + (p-1)y + z &= 3 \\ x + 2y + z &= p \end{aligned}$$
for all real parameters p.

4. Find all solutions to

$$\begin{aligned} x+2y-3z+2t &= 4 \\ 2x-y-z-t &= -2 \\ 5x-5z &= 0 \\ -5y+5z-5t &= -5 \end{aligned}$$

5. Solve

$$\begin{aligned} x-y+2z+t &= 3 \\ 2x-3y+4z-t &= 2 \end{aligned}$$

6. Solve

$$\begin{aligned} x+y-2z+t &= 0 \\ 3x-y-t &= 3 \\ x-y+z-t &= 0 \end{aligned}$$

7. Find A^{-1} for

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then use it to find solution to

$$\begin{aligned} 2x+y+z &= 3 \\ x+3z &= -7 \\ 2x+y &= 1 \end{aligned}$$

and a solution to

$$\begin{aligned} 2x+y+z &= 0 \\ x+3z &= 3 \\ 2x+y &= -1 \end{aligned}$$

2.4 Square matrices – intro

1. Find a matrix X fulfilling

$$X\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix} = 2\begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}$$

2. Find all matrices X such that

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} X + \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 11 & 15 \end{pmatrix}$$

2.5 Square matrices – determinants

1. Compute

$$\det\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 2 & -2 & -2 \end{pmatrix}$$

2. Find all $\alpha \in \mathbb{R}$ for which the matrix

$$\begin{pmatrix} \alpha & 3 \\ 1 & \alpha \end{pmatrix}$$

is singular.

3. Use the definition of determinant to compute

$$\det\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 5 \\ 2 & 1 & 4 & 4 \end{pmatrix}$$

4. Find the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 5 & 7 & 10 \\ 1 & 2 & 3 & 6 & 7 \\ 1 & 1 & 3 & 5 & 5 \\ 1 & 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

5. Use the Cramer rule to solve

$$\begin{aligned} x+2y-z &= 0 \\ -x+y+2z &= 1 \\ 2x+y-z &= 1. \end{aligned}$$

6. Compute

$$\det\begin{pmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 10 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

2.6 Eigenvalues

1. Write down the characteristic polynomial of

$$A = \begin{pmatrix} 0 & 5 & 3 \\ -1 & 2 & -1 \\ 1 & -5 & -2 \end{pmatrix}$$

and verify that $\lambda_1 = -3$, $\lambda_2 = 2$, and $\lambda_3 = 4$ are its roots. Then compute the eigenvector corresponding to λ_2

2. Compute all eigenvalues of

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

and find the corresponding eigenvectors.

3. Find all eigenvalues of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and find the corresponding eigenvectors including the generalized one.

4. Find all eigenvalues and eigenvectors of

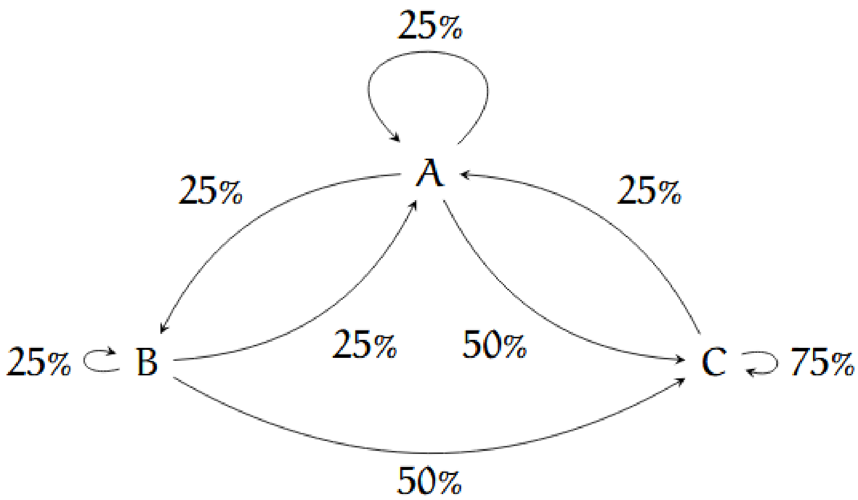
$$A = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 1 & -5 \\ 2 & 1 & 3 \end{pmatrix}$$

5. A rental car company has three basis in three different cities A, B, C. The customers can pick up and return cars at any station. For example, a customer could pick up his car at A and return it in C.

We assume that there is only one car type available and that all customers rent their cars for one fixed time period (say one day). The graph below shows the proportions of cars that remain at a station or are transferred to other station over a day.

(a) Find an appropriate matrix $P \in \mathbb{R}^3$ that describes this transition process, i.e., a matrix P such that $P\nu_0$ gives the distribution of cars after one day assuming ν_0 is the initial distribution.

(b) How should the cars be distributed initially such that the distribution does not change over time? (i.e., find a distribution ν such that $P\nu = \nu$). Is it possible?



2.7 Square matrices – definiteness

1. Write the quadratic form Q whose matrix is

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 2 \\ -2 & 2 & 5 \end{pmatrix}$$

2. Write the matrix corresponding to the quadratic form

$$Q(x,y,z,t) = 2x^2+3y^2-xy-zt+2xt+5yz-4xz-6yt+z^2$$

3. Decide about the definiteness of the following forms

$$Q(x,y,z) = x^2+2xy+y^2-z^2+2(xz+yz)$$
$$Q(x,z,z) = 2x^2+2xy+2y^2+2xz+2z^2$$

4. Decide about the definiteness of the following matrices

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 0 \\ 3 & 5 & 6 \\ 0 & 6 & -4 \end{pmatrix}$$

2.8 Revision

1. Find all vectors v belonging to $\operatorname{span}\{(1,-1,2),(1,2,0)\} \cap \operatorname{span}\{(0,-1,-1),(-1,2,1)\}$.

2. Find coordinates of $(1,-1,1)$ with respect to a basis $(-1,1,0)$, $(0,-1,1)$ and $(1,0,-1)$.

3. Are vectors $(2,1,3)$, $(-1,2,1)$, $(0,-1,-1)$ linearly independent?

4. Decide about the rank of

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 5 & 7 & -4 \\ 1 & 1 & -2 \\ 3 & 4 & -3 \end{pmatrix}$$

5. The eigenvalues of

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

are 2 and 3. Find the eigenvector and the generalized eigenvector corresponding to $\lambda = 2$.

6. Let there is a system of a thousand equations and thousand unknowns whose matrix is regular. Assume that the vector of the right hand side is the same as the column vector corresponding to the x_{485} variable. What is the solution to that system? Justify your claim.

7. Find the eigenvalues to

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$$

8. What is the matrix associated to the quadratic form

$$Q(x,y,z) = x^2+2xy+z^2+2y^2-zy$$

Decide about its definitness.

9. Is there any solution (x,y,z) to an equation

$$x^2+2xy+2yz-3zx+2z^2+x^2 = -2?$$

3 Functions

3.1 Mapping, basic notion

1. Which of these subsets are mappings?
   - $f = \{\langle 1,5\rangle, \langle 2,4\rangle, \langle 1,3\rangle\}$,
   - $g = \{\langle 1,2\rangle, \langle 5,3\rangle, \langle 10,1\rangle\}$,
   - $h = \{\langle 3,3\rangle, \langle 4,3\rangle, \langle 7,7\rangle, \langle 10,3\rangle\}$.

   If f, g, or h is a mapping, determine its domain and range.

2. Let $A = \{1,2,3,4\}$ and $B = \{2,5,8\}$. Consider a mapping $f : A \to B$ which is given as
   $$\{\langle 1,5\rangle,\ \langle 3,2\rangle,\ \langle 2,2\rangle\}.$$
   Write down Dom$f$, Ran$f$ and decide whether f is injection or surjection. Is there f^{-1}? If yes, find it and determine its range and domain. Further, let $g : B \to A$ be defined as $g = \{\langle 2,1\rangle, \langle 8,4\rangle\}$. Determine $g \circ f$.

3. Let $A = \{2,4,5,6\}$ and $B = \{1,2,3,5\}$. Let $f : A \to B$ be given as
   $$\{\langle 2,2\rangle, \langle 4,2\rangle, \langle 5,1\rangle, \langle 6,5\rangle\}.$$
   Determine the range and domain of f. Is there f^{-1}? Find $f(\{2,4\})$, $f(\{2,5,6\})$, $f^{-1}(\{1\})$, $f^{-1}(\{1,2\})$ and $f^{-1}(\{2,3,5\})$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given as
   $$f(x) = \sqrt{x^2 - x + 2}$$
   and let $g = x + 3$. Find $f(g(x))$ and $g(f(x))$.

5. Find f^{-1} for a function $f(x) = \frac{x+3}{2x-1}$, $x \in \mathbb{R} \setminus \{\frac{1}{2}\}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) = x^2 - 4x + 5$ with domain $x \in (2, \infty)$. Find f^{-1}.

7. Show that $f(x) = x + 2$, $x \in [3, \infty)$ is bounded from below and not from above.

8. Let $f : \mathbb{N} \to \mathbb{N}$ be given as $f(n) = 2n$. Is this function bounded from below and from above? Justify your claim. Can you give an example of function $g : \mathbb{N} \to \mathbb{N}$ which is not bounded from below nor above?

3.2 Real functions – intro

1. Find maximal domain of functions
   - $f(x) = \sqrt{\frac{x+1}{x-1}}$,
   - $g(x) = \frac{1}{\sqrt{x^2+5x+4}}$.

2. Find the maximal domain of functions
   - $f(x) = \frac{1}{(\log(\sin x))^2}$,
   - $g(x) = \frac{\sqrt{x}}{e^x}$.

3. Find the maximal domain of functions
   - $f(x) = \sqrt{x^2 + 6x + 3} + \sqrt[3]{x+1}$,
   - $g(x) = 5^{x^2 + \ln x}$

4. Decide about the parity of the following functions
   - $f(x) = \frac{\sin x}{x^3 + x}$
   - $g(x) = \sqrt{x^2 + 1} \cos x$

   and justify your answer.

5. Decide about the parity of the following functions
   - $f(x) = \frac{x+1}{x-1}$
   - $g(x) = \frac{x^2+1}{x + \sin x}$

   and justify your answer.

6. Use the definition of continuity to show that
   (a) $f(x) = 2 + \frac{x}{2}$ is continuous at a point $x_0 = 1$,
   (b) $g(x) = x^2$ is continuous at a point $x_0 = 0$,
   (c) $h(x) = x$ is continuous on the whole real line.

7. Sketch a graph of a function
   $$f(x) = \mathrm{sgn}(\sin x)$$
   and decide about monotonicity, periodicity, range, domain, boundedness, and continuity. Here sgn is defined as follows
   $$\mathrm{sgn}(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ if } x < 0. \end{cases}$$

8. Prove that the function f from the previous exercise is continuous in every point of a set $\mathbb{R} \setminus \{x = k\pi,\ k \in \mathbb{Z}\}$ and discontinuous everywhere else.

3.3 Polynomials

1. Find all (complex) roots of $p(x) = 2x^2 + 8x + 16$.
2. Find all roots of $p(x) = x^3 + 3x^2 - 10x - 24$.

3.4 Real powers

1. Decide, which of the two numbers is higher
   - $5^{1/4}$, $5^{1/2}$,
   - $\left(\frac{2}{3}\right)^2$, $\left(\frac{2}{3}\right)^{2.2}$,
   - $(\sqrt{2})^{-1}$, $(\sqrt{2})^{-0.66}$.

2. Find all x satisfying
   $$\sqrt{x - x^2 + 12} < \sqrt{7 - 3x}.$$

3.5 Exponential functions and logarithms

1. Solve
$$\frac{27^{3x-2}}{243} = 81^{3x-7}.$$
2. Solve
$$4^x + 2^{x+2} = 5.$$
3. Solve
$$\log_4(x^2-9) - \log_4(x+3) = 3.$$
4. Find all real numbers x satisfying
$$e^x < c.$$

3.6 Goniometric functions

1. Solve $\sin x = \frac{1}{2}$.
2. Solve $4\sin^2 x - 4\sin x + 1 = 0$.
3. Find all $x \in \mathbb{R}$ satisfying
$$\cos x > -\frac{1}{2}.$$

3.7 Limits 1

1. Solve
$$\lim_{x\to 0} \frac{x^2-1}{2x^2-x-1}$$
2. Solve
$$\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1}$$
3. Solve
$$\lim_{x\to\infty} \frac{(x+1)(x+2)(x+3)(x+4)(x+5)}{(5x+1)^5}$$
4. Solve
$$\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$$
5. Solve
$$\lim_{x\to\pi} \frac{\sin x}{x-\pi}$$
6. Solve
$$\lim_{x\to 0} \frac{5^x - e^x}{x}$$
7. Solve
$$\lim_{x\to 3} \frac{x^3-2x^2-2x-3}{x^2-6x+9}$$
8. Solve
$$\lim_{x\to -\infty} \frac{x^4+4x+1}{x^3-x+1}$$
9. Solve
$$\lim_{x\to 0} x\cot(3x)$$
10. Solve
$$\lim_{x\to 0} (x+e^x)^{\frac{1}{x}}$$

3.8 Limits 2

1. Solve
$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{3x}.$$
2. Solve
$$\lim_{x\to\infty} \left(\frac{3x+7}{3x}\right)^{x-1}.$$
3. Solve
$$\lim_{x\to\infty} \frac{\sqrt{x}-2}{x}.$$
4. Solve
$$\lim_{x\to\infty} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$
5. Solve
$$\lim_{x\to\infty} (x+1)\left(\log(x+1)-\log x\right).$$
6. Solve
$$\lim_{x\to 0} \frac{\sin x}{x^2}.$$
7. Solve
$$\lim_{x\to 0} \frac{1-\cos(2x)}{x\sin x}.$$
8. Solve
$$\lim_{x\to\infty} \left(\frac{3x-2}{3x+1}\right)^{2x}.$$

3.9 Derivatives

1. Solve $\left(2x^3+\sqrt{x}-\frac{1}{x^2}\right)'$
2. Solve $\left(\frac{1}{3}x^4+x-1\right)'$
3. Solve $\left((x^2+2x)^5\right)'$
4. Solve $\left(x^3+\sin x+e^{x^2}\right)'$
5. Solve $\left(\frac{x^2\sin x}{2+\cos x}\right)'$
6. Solve $\left(\sqrt{x^2+3}\right)'$
7. Solve $\left(x^2\sqrt{x+1}\right)'$
8. Solve $\left(\frac{1}{\sqrt[3]{3x+5}}\right)'$
9. Solve $\left(\sqrt[3]{x+4}-\frac{1}{x^2+3}\right)'$
10. Solve $(\log 5x)'$
11. Solve $\left(3^{x^2+3x}\right)'$
12. Solve $\left(\frac{e^{x^2}}{x^2\arctan x}\right)'$

13. Solve $\left(e^{x\sin x}\right)'$

14. Solve $\left(x^{3}\frac{x^2+e^{\sin x}}{x^2+1}\right)'$

3.10 Tangent lines

1. Find a tangent line to a graph of a function $f(x) = x^2+6x+1$ which passes through a point $A=[2,17]$.

2. Find a tangent line to a graph of a function $f(x) = \sqrt{4-x^2}$ which passes through a point $A=[1,\sqrt{3}]$.

3.11 Extremes and monotonicity

1. Find the local extremes of $f(x) = x^3-3x^2+1$.

2. Find the global extremes of $f(x) = x^2+\frac{1}{x^2}$ on $[-2,2]$.

3. Find the local maximum and minimum of $f(x) = \frac{1}{2}x+\sin x$, $\mathrm{Dom}f=[0,2\pi]$.

4. The total revenue TR depends on the production Q and prize P as $TR = Q\cdot P$. Let the prize be a function of the demand, i.e., $P = D(Q)$. Find the maximum of TR if the demand is described as $D(Q) = 60-3Q$.

5. The profit PR is computed as $PR = TR - TC$ where TC is the total cost. Let TC depends on the production Q as Q^2+5Q+4 and let TR be given as $45Q-Q^2$. Find the maximum of PR.

6. Find the global extremes of $f(x) = -x^3+6x^2+8$ on $[-1,7]$.

7. Find the global extremes of $f(x) = \sqrt{x}+\frac{4}{x}-2$ on $(0,7]$.

8. Determine the intervals of monotonicity of $f(x) = \frac{2x-1}{3x-1}$.

9. Determine the intervals of monotonicity of $f(x) = 1-\sqrt{x^3-1}$.

10. Determine the intervals of monotonicity of $f(x) = 3x^3-6x^2+4|x|$.

3.12 Curvature

1. Decide about the curvature and points of inflection of $f(x) = \frac{2x}{x^2+1}$.

2. Decide about the curvature and points of inflection of $f(x) = \frac{x^2}{x^2-16}$.

3.13 The course of a function I

1. Examine the course of $f(x) = x^3+2x$.

2. Examine the course of $f(x) = -x^4+6x^2-5$.

3. Draw the graph of the function f with the following properties: $\mathrm{Dom}f = \mathbb{R}\setminus\{\pm 4\}$, f is even, f is bounded, $\lim_{x\to\infty} f(x) = 3$, $f'(0) = 0$, $f(2) = 1$, $f'(x) < 0$ for $x > 4$.

4. Examine the course of $f(x) = e^{x-x^2}$.

5. Draw the graph of the function f with the following properties: $\mathrm{Dom}f = \mathbb{R}\setminus[-1,1]$, f is odd, f is concave on $(-\infty,-1)$, $f'(2) = 0$, $\lim_{x\to 1+} f(x) = \infty$.

6. Examine the course of $f(x) = \frac{x^3}{x^2-1}$.

7. Examine the course of $f(x) = \sin x + \cos x$.

8. Examine the course of $f(x) = (x^2+1)3x$.

3.14 l'Hospital rule

1. Solve $\lim_{x\to 0}\frac{\arctan x}{x^2-x}$

2. Solve $\lim_{x\to 1}\frac{\log^2 x}{x-1}$

3. Solve $\lim_{x\to 0+} x\log x$

4. Solve $\lim_{x\to\infty} x^{\frac{1}{x}}$

5. Solve $\lim_{x\to 1}\left(\frac{1}{x\log x}-\frac{x}{\log x}\right)$

6. Solve $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

3.15 The course of a function II

1. Examine the course of $f(x) = 2x^2-\log x$

2. Examine the course of $f(x) = x^2e^{-x}$

3.16 Taylor polynomial

1. Write the second-degree Taylor polynomial for $f(x) = \arctan x$ at $x_0 = 1$.

2. Find the Taylor polynomial of degree 4 for the function $f(x) = \log x$, centered at $x_0 = 2$.

3. Use the third-degree Taylor polynomial to deduce the approximate value of $\sqrt{5}$. What is the maximal error you made?

4. Find the value of $\sqrt{10}$ with error not exceeding 0.0001.

4 Function of multiple variables

4.1 Introduction

1. Determine and sketch the domain of a function $f = x + \sqrt{y}$.

2. Determine and sketch the domain of a function $f = \sqrt[3]{\frac{1}{xy}}$.

3. Let $f(x,y) = 4x^2 + y$. Write a function
$$g(t) = f(2+t, 3+2t)$$
and draw its graph.

4. Determine and sketch the domain of a function $f(x,y) = \sqrt{|x| + |y| - 2}$.

5. Find and sketch the domain of a function
$$f(x,y) = \sqrt{16 - 4x^2 - y^2}.$$
What is the range of this function?

6. Determine and sketch the domain of a function $f(x,y) = \log(x \log(y-x))$.

7. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x,y) = xy$.

8. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x,y) = |x| + y$.

9. Sketch contour lines at heights $z_0 = -2, -1, 0, 1, 2$ of a function $f(x,y) = (x+y)^2$.

10. Find $g(t) = f(t, t+1)$ where $f(x,y) = x^2 + 2xy$. Draw the graph of $g(t)$.

11. Find $g(t) = f(t, t^2)$ where $f(x,y) = x - y^2$. Draw the graph of $g(t)$.

4.2 Derivatives – intro

1. Compute the derivative of $f(x,y) = \frac{x^2+1}{y^2+1}$ with respect to direction $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in a point $(1,0)$.

2. Compute $Df((2,1), (\cos\alpha, \sin\alpha))$ where
$$f(x,y) = x + xy$$
and $\alpha \in [0, 2\pi)$ is a parameter.

3. Compute ∇x^y.

4. Compute
$$\nabla x \sin(x+y).$$

5. Compute all first order partial derivatives of
$$f(x,y,z) = e^y x^2 + \frac{y}{z}$$

6. Compute all partial derivatives of the first and second order of a function
$$f(x,y) = xy + \frac{x}{y}.$$

7. Compute all second derivatives of
$$f(x,y) = \arctan\frac{x+y}{1-xy}.$$

8. Compute the first and the second order partial derivatives of a function
$$f(x,y) = \frac{y}{\sin x}.$$

9. Determine all first order derivatives of
$$f(x,y) = (3x+2y)^{\log y}.$$

10. Given the following information use the Chain rule to determine $\frac{\partial f}{\partial t}$:
$$f(x,y) = \cos(yx^2),\ x = t^4 - 2t,\ y = 1 - t^6.$$

11. Compute $\frac{\partial f}{\partial t}$ if
$$f(x,y,z) = \frac{x^2 - z}{y^4},\ x = t^3 + 7,\ y = \cos(2t),\ z = 4t.$$

12. Use the differential of an appropriate function in order to compute
$$\sqrt{8.5} \cdot \log(1.05).$$

13. Determine the tangent plane to
$$f(x,y) = x^2 - 2xy - 3y^2$$
at point $A = (-1, 1)$.

14. Write the second degree Taylor polynomial at point $(1,3)$ of a function
$$f(x,y) = x^2 y + y^2.$$

15. Use the second degree Taylor polynomial to determine an approximate value of
$$\sqrt{20 - (1.9)^2 - 6.6}.$$

4.3 Implicitly given functions

1. Show that there is a function $y(x)$ determined by
$$x^2 + y^2 + xy = 3$$
in the vicinity of $(1,1)$. Compute the first and the second derivative of $y(x)$ at $x = 1$.

2. Compute y' and y'' of a function given by the equality
$$(x^2+y^2)^2 - 3x^2 y - y^3 = 0.$$
(Tacitly assume the function exists and therefore there is no need to use the implicit function theorem).

3. Show that there is a function $y(x)$ determined by the relation
$$2^{xy} - 3^{x+y} = -23$$
in the neighbourhood of $(1,2)$. Compute y' and y'' at $x = 1$.

4. Show that there is a function $y(x)$ determined by

$$y - \frac{1}{2}\sin y = x$$

in the neighborhood of $\left(\frac{\pi-1}{2}, \frac{\pi}{2}\right)$. Decide, whether $y(x)$ is convex or concave in some neighborhood of the given point.

5. Consider an equation

$$-x^2 + y^2 - 2xy + y = 0.$$

Identify all $x_0 \in \mathbb{R}$ where there is a function $y(x)$ defined on their neighborhood and which are stationary points of $y(x)$. Decide, whether there are local extremes at the stationary points.

4.4 Extremes

1. Find local extrema of the following functions

$$f(x,y) = x^2 - xy + y^2 - 2x + y.$$

2. Examine local extrema of

$$f(x,y) = x^3 + x^2y - y^2 - 4y$$

3. Find three positive numbers x, y, z whose sum is 10 such that x^2y^2z is a maximum.

4. Find and identify local extrema of

$$f(x,y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$$

5. Examine local extrema of

$$f(x,y) = e^{-x^2-y^2}$$

6. For a rectangular solid of volume 1000 cubic meters, find the dimensions that will minimize the surface area.

7. Find all local extrema of

$$f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$$

8. Find local extreama of

$$f(x,y) = (\sin x)^2(y-1)^2$$