Extremes

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Extremes: 1D remainder

Definition

Let $f : \mathbb{R} \to \mathbb{R}$. We say that f attains its *local maximum* (resp. minimum) at a point x_0 if there exists $\varepsilon > 0$ such that $f(x_0) > f(x)$ (resp. $f(x_0) < f(x)$) for every $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\}$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be of class C^1. A **stationary point** of f is a point x_0 such that $f'(x_0) = 0$.

Lemma

Lef $f \mathbb{R} \to \mathbb{R}$ be of class C^2 and let x_0 be its stationary points. Then if $f''(x_0) > 0$, f attains its local minimum at x_0 and if $f''(x_0) < 0$, f attains its local maximum at x_0.

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Example

- Find the local extrema of $f(x) = \sin x + \frac{1}{2}x$.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ attains its *(global) maximum* at x_0 if $f(x_0) \geq f(x)$ for all $x \in \mathrm{Dom}\ f$. A *(global) minimum* is defined respectively.

Lemma

Let f be a continuous function defined on $[a, b] \subset \mathbb{R}$. Then f attains its maximum and minimum on $[a, b]$.

Example

- Find the global extrema of $f(x) = x^2 e^x$.
- Find the global extrema of $f(x) = x^3 - 12x$ on $[-3, 5]$

Monopoly competition: The total profit of a company is given as $TP = TR - TC$, where TR stands for total revenues and TC stands for total costs. Let Q stands for the quantity of produced goods. Naturally, $TR = P * Q$ where P is the market price of the product.
Assume $TC = 500\,000 + 400Q + 0.04Q^2$ (fix costs, variable costs, ...).

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Monopoly competition: $MR := TR'(Q) < P(Q)$

Extremes, multi-D reminder

Definition

Let $f : M \subset \mathbb{R}^n \mapsto \mathbb{R}$. We say that f attains a *local maximum* at a point $x_0 \in M^0$ if there is $r > 0$ such that $f(x_0) \geq f(x)$ for all $(x) \in B_r(x_0)$. A *Local minimum* is defined analogously.

Definition

Let $f : \mathbb{R}^d \to \mathbb{R}$ be of class C^1. A **stationary point** of f is a point x_0 such that $\nabla f(x_0) = 0$.

Lemma

Let $f \in C^2$ and let x_0 be its stationary point. Then

- *if $\nabla^2 f(x_0)$ is positive-definite, then f has a local minimum at x_0,*
- *if $\nabla^2 f(x_0)$ is negative-definite, then f has a local maximum at x_0,*
- *if $\nabla^2 f(x_0)$ is indefinite, then there is no extreme at x_0,*
- *otherwise, we do not know anything.*

Example

- Examine the local extrema of

$$f(x,y) = y^3 + x^2 - 6xy + 3x + 6y - 7$$

- Examine the local extrema of

$$f(x,y) = x^2y^2 - x^2 - y^2$$

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ attains its *(global) maximum* at x_0 if $f(x_0) \geq f(x)$ for all $x \in \mathrm{Dom}\ f$. A *(global) minimum* is defined respectively.

Example

- A company manufactures two products A and B that sell for \$10 and \$9 per unit respectively. The cost of producing x units of A and y units of B is

$$400 + 2x + 3y + 0.01(3x^2 + xy + 3y^2).$$

Find the values of x and y that maximize company's profit.

Definition

A set $M \subset \mathbb{R}^n$ is *convex* if for every $x, y \in M$ and every $\lambda \in (0,1)$ it holds that

$$\lambda x + (1-\lambda)y \in M.$$

Definition

Let $\mathrm{Dom}\ f \subset \mathbb{R}^n$ be a convex set. We say that f is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in \mathrm{Dom}\ f$ and $\lambda \in (0,1)$. The function is strictly convex, if

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y).$$

The function f is (strictly) concave if $-f$ is (strictly) convex.

Several observation

- The second gradient is positive (negative) definite – the function is strictly convex (concave).
- The function is convex on its domain – every local minimum is a global minimum.
- The function is concave on its domain – every local maximum is a global maximum.

Global extremes with respect to a set

Lemma

Let $M \subset \mathbb{R}^n$ *and let* $f : M \to \mathbb{R}$. *Then* f *attains its* **minimum on** M *at point* $(x_0, y_0) \in \mathbb{M}$ *if*

$$\forall (x, y) \in M, \; f(x_0, y_0) \leq f(x, y).$$

Similarly, f *attains its* **maximum on** M *at point* $(x_0, y_0) \in \mathbb{M}$ *if*

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Lemma

Let $M \subset \mathbb{R}^n$ be a bounded and closed set and let $f : M \to \mathbb{R}$ be a continuous function. Then f attains its minimum and maximum on M.

Examples

- Find the maximum and minimum of
$$f(x,y) = x^2 + y^2 - 2xy$$
on the set $M = \{(x,y) \in \mathbb{R}^2,\ x \in (-1,3),\ \ y \in (0,2)\}$.

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on the set $M = \{(x, y) \in \mathbb{R}^2,\ x \in (-1, 3),\ y \in (0, 2)\}$.
- Solve: To reduce shipping distances between the manufacturing facilities and a major consumer, a Korean computer brand, Intel Corp. intends to start production of a new controlling chip for Pentium III microprocessors at their two Asian plants. The cost of producing x chips at Chiangmai (Thailand) is
$$C_1 = -0.002x^2 + 50x + 500$$
and the cost of producing y chips at Kuala-Lumpur (Malaysia) is
$$C_2 = 0.005y^2 + 4y + 275.$$
The Korean computer manufacturer buys them for $150 per chip. Find the quantity that should be produced at each Asian location to maximize the profit if the maximum delivered amount is 50 000 and the factory in Chiangmai is able to produce at most 20 000 chips.

- Find the maximum and minimum of

$$f(x, y) = (x^2 + y)e^y$$

on the set

$$M = \{(x, y) \in \mathbb{R}^2,\ y \geq \frac{1}{3}x,\ y \leq 3x,\ y \leq 5 - x\}.$$

- Find the extreme values of

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

on the region described by the inequality

$$x^2 + y^2 = 16.$$

Theorem (The Lagrange multipliers)

Let $f : \mathrm{Dom}\, f \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^1 *function defined on a neighborhood of*

$$M = \{x \in \mathbb{R}^n, g(x) = 0\}$$

where g *is a* C^1 *function. If there is an extreme of* f *with respect to the set* M *at* $x_0 \in \mathbb{M}$*, then there exists* $\lambda \in \mathbb{R}$ *such that*

$$\nabla f(x_0) + \lambda \nabla g(x_0) = 0,$$

or $\nabla g(x_0) = 0$*.*

Exercises

- Find extremes of

$$f(x,y) = x^2 + y^2 - 12x - 16y$$

  on

$$M = \{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 \leq 25,\ x \geq 0\}.$$

-

- Find the maximum and minimum values of

$$f(x,y,z) = y^2 - 10z$$

  subject to the constraint

$$x^2 + y^2 + z^2 = 36.$$

Theorem (The Lagrange multipliers - two constraints)

Let $n \geq 3$, $f : \mathrm{Dom}\, f \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^1 *function defined on a neighborhood of*

$$M = \{x \in \mathbb{R}^n,\ g(x) = 0,\ h(x) = 0\}$$

where $g, h : \mathbb{R}^n \to \mathbb{R}$ *are* C^1 *functions. If there is an extreme of* f *with respect to the set* M *at* x_0, *then there exists* λ, $\mu \in \mathbb{R}$ *such that*

$$\nabla f(x_0) + \lambda \nabla g(x_0) + \mu \nabla h(x_0) = 0,$$

or $\nabla g(x_0)$ *and* $\nabla h(x_0)$ *are linearly dependent.*

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Example

- Find the maximum and minimum values of

$$f(x, y, z) = 3x^2 + y$$

subject to the constraints

$$4x - 3y = 9 \quad \text{and } x^2 + z^2 = 9.$$

Another exercise

- Find the maximum and the minimum of

$$f(x,y,z) = xy + xz + yz$$

on the set

$$M = \{(x,y,z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 2,\ z \leq 1\}.$$

Tricks and traps

- Find the maximum and minimum of $f(x,y) = x + y$ on $M = \left\{(x,y) \in \mathbb{R}^2,\ x^3 + y^3 - 2xy = 0,\ x \geq 0,\ y \geq 0\right\}$.
- Find the maximum and minimum of $f(x,y) = -y + xz$ on $M = \left\{(x,y,z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 1,\ x^2 + y^2 = 1\right\}$.
- Find the maximum and minimum of $f(x,y) = x^2 + y^2$ on $M = \left\{(x,y) \in \mathbb{R}^2,\ \frac{x}{2} + \frac{y}{3} = 1\right\}$.

Applications

- Suppose you are running a factory producing some sort of widget that requires steel as a raw material. Your costs are predominantly human labor, which is \$20 per hour for your worker, and the steel itself, which runs for \$170 per ton. Suppose your revenue R is loosely modeled by the following equation

$$R(h, s) = 200h^{2/3}s^{1/3}$$

  where h represents hours of labor and s represents tons of steel. If your budget is \$20 000, what is the maximum possible revenue?
- The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize the cost.