1 Linear algebra

1.1 Vectors

1. Let $u = (-1, 5)$, $v = (2.7, 3.8)$, $w = (4.2, -6)$. Compute $u + v$, $u - v$, $v - u$, $v - w$, $u + 2w$, $3u - v$, $u + v + w$, $3u - 2v - 5w$, $\frac{1}{2}w + \frac{3}{4}v - \frac{5}{2}u$.

2. Are vectors
$$\begin{aligned} v_1 &= (1, 1, 1, 2) \\ v_2 &= (1, 2, -1, 1) \\ v_3 &= (0, 1, 1, 2) \end{aligned}$$
linearly independent?

3. Check if the vector $u = (2, 0, -1)$ is a linear combination of $v = (6, -2, 4)$ and $w = (-3, -1, 2)$.

4. Find the value of $k \in \mathbb{R}$ such that the vector $u = (k, 4, k)$ is a linear combination of $v = (-1, 2, 2)$ and $w = (4, 2, 1)$.

5. Polynomial $P(x) = x^2 + 3x + c$, $c \in \mathbb{R}$ belongs to a linear span of $Q(x) = 2x^2 - 1$ and $R(x) = x + 2$. Determine the value of c.

6. Find λ_1, λ_2 and λ_3 such that
$$\lambda_1(2, 1, 3) + \lambda_2(0, -2, 0) + \lambda_3(1, 4, 2) = (9, 9, 14)$$
(remark: λ_1, λ_2, λ_3 are coordinates of $(9, 9, 14)$ with respect to the basis
$$\{(2, 1, 3),\ (0, -2, 0),\ (1, 4, 2)\}.)$$

7. Let
$$u = (2, 4, 6),\ v = (-1, -2, -3),\ w = (-2, -4, -6).$$
Is there a vector z such that $u \notin \operatorname{span}\{v, w, z\}$?

8. How the dimension of
$$\operatorname{span}\{(1, 1, 1), (2, 2, 2), (3, 3, k)\}$$
depends on k?

9. Determine, whether b is a linear combination of a_1, a_2 and a_3 where
$$\begin{aligned} a_1 = (1, -2, 2),\ a_2 = (0, 5, 5), \\ a_3 = (2, 0, 8),\ b = (-5, 11, 8). \end{aligned}$$

10. Let
$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Show that $\operatorname{span}\{M_1, M_2, M_3\}$ is a space of all symmetric matrices.

1.2 Matrices – intro

1. Let
$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 \end{pmatrix},\ C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$
Compute (if possible)
$$2A + BC,\ AB + C,\ CB - A,\ 2BA + C,\ B\left(C + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$$

2. Is there $\alpha \in \mathbb{R}$ such that the matrix
$$A = \begin{pmatrix} 1 & \alpha & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
has a rank 3?

3. Do the vectors
$$(0, 0, -2),\ (0, -3, 8),\ (4, -1, -5)$$
span $\mathbb{R}^3$?

4. Determine the rank of the matrix $A = \begin{pmatrix} 3 & 2 & 3 \\ -2 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix}$ by using the definition of rank.

1.3 Matrices – the Gauss elimination method

1. Determine the rank of
$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 2 & 4 & 5 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 2 & 5 & -1 \end{pmatrix}$$

2. Solve
$$\begin{aligned} 3x + y - z &= 1 \\ x - y + z &= -3 \\ 2x + y + z &= 0 \end{aligned}$$

3. Solve
$$\begin{aligned} 2x + 5y &= 9 \\ x + 2y - z &= 3 \\ -3x - 4z + 7z &= 1 \end{aligned}$$

4. An amount of \$65,000 is invested in three bonds at the rates of 6%, 8% and 10% per annum respectively. The total annual income is \$4,800. The income from the third bond is \$600 more than that from the second bond. Determine the price of each bond.

5. Find all solutions to
$$\begin{aligned} 2x - 3y + z &= 2 \\ -x + 2y - z &= -2 \\ 3x - 4y + z &= 2 \end{aligned}$$

1.4 Square matrices – intro

1. Verify that

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}.$$

2. Determine

$$\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}^{-1}$$

3. Compute

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}^{-1}$$

4. Is the matrix X solving

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} X + \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

regular or singular?

1.5 Square matrices – determinants

1. Compute

$$\det \begin{pmatrix} \lambda - 2 & 3 \\ 1 - \lambda & 2 \end{pmatrix}$$

where $\lambda \in \mathbb{R}$.

2. Compute

$$\det \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

3. Compute

$$\det \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 3 & 3 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

4. Use the Cramer rule to solve

$$\begin{aligned} 2x - y + z &= 3 \\ 3x + 2y + z &= 7 \\ -x - y - 2z &= -6 \end{aligned}$$

5. Compute

$$\det \begin{pmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 & 3 \\ -2 & 0 & 0 & 0 & 3 \\ 1 & -1 & 1 & -1 & 3 \end{pmatrix}$$

1.6 Square matrices – eigenvalues

1. Determine the eigenvalues and eigenvectors (including the generalized ones) of a matrix

$$A = \begin{pmatrix} -5 & -3 & -1 \\ 16 & 9 & 3 \\ -2 & -1 & 1 \end{pmatrix}$$

2. Find all eigenvectors and eigenvalues of

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

3. Find a matrix whose characteristic polynomial is $-\lambda^3 + 2\lambda^2 + \lambda - 2$.

4. Find eigenvalues (not eigenvectors) of

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 4 & 3 & -2 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 2 & 3 \end{pmatrix}.$$

5. Find all vectors (including the generalized) of

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -4 & 4 \end{pmatrix}.$$

1.7 Square matrices – quadratic forms and definiteness

1. Write the quadratic form Q whose corresponding matrix is
   - $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
   - $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix}$
2. Write the symmetric matrix corresponding to
   - $Q(x, y) = x^2 + 3xy - y^2$
   - $Q(x, y, z) = 2x^2 - 5xy + 3yz - 2z^2$
   - $Q(x, y, z) = x(x + 2y) + (3x - y)(2x - 2y)$
3. Decide about the definiteness of the following matrices:
   - $\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$
   - $\begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 2 \end{pmatrix}$
   - $\begin{pmatrix} -2 & -2 & -1 & 1 \\ -2 & -4 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}$

2 Sequences and series

2.1 sequences – basic properties

1. Find an explicit formula of the sequence given as
$$a_1 = 1,\ a_n = n + a_{n-1} \text{ for } n \geq 2$$
and decide about the monotonicity and boundedness of the sequence.

2. Determine, whether the sequence
$$a_n = \frac{n+2}{2n-1}$$
is bounded or monotone. Justify your answer.

3. Give a guess of an explicit formula for a sequence whose first terms are
$$\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \dots\right\}.$$

4. Find an explicit formula for a sequence given as
$$a_1 = \frac{1}{2},\ a_{n+1} = a_n + \left(\frac{1}{2}\right)^{n+1} \text{ for } n \geq 1.$$

5. Below you can find 6 sequences. Which of them are monotone? And which of them are bounded?
$$a_n = \frac{1}{n^2},\ b_n = 2^n,\ c_n = \frac{3^n}{n^3}$$
$$d_n = \frac{2n^3+1}{5n^3+3n^2+2},\ e_n = (0,9)^n,\ f_n = \frac{\cos(n\pi)}{n^2}$$

2.2 sequences – limits

1. Solve
$$\lim \left(\sqrt{2n} - \sqrt{2n-1}\right)\sqrt{n}$$

2. Solve
$$\lim \frac{\sqrt{n} + n^{\frac{1}{3}}}{n^{\frac{1}{2}} - 1}$$

3. Solve
$$\lim \frac{(n+1)^4}{(n+\sqrt{n})^3}$$

4. Solve
$$\lim \left(\frac{2n-3}{2n}\right)^n$$

5. Solve
$$\lim \frac{\sqrt[3]{n}^4 + n^2 2^n}{n - 1 + e^n}$$

6. Solve
$$\lim \frac{\sqrt[3]{n^2}}{n+1} \sin n!$$

7. Solve
$$\lim \frac{3n^6 + 4n - 3}{(2n+3)^6}$$

8. Solve
$$\lim \sqrt{n^4 - 5n} - n^2$$

9. Solve
$$\lim \frac{(-1)^n(\sqrt{n^2+1} - 1)}{n}$$

10. Solve
$$\lim \left(\frac{2n-1}{2n+1}\right)^n$$

11. Solve
$$\lim \sqrt{n^2+3n} - n$$

12. Solve
$$\lim \frac{(2n+1)^2(n-2)^3}{(n+1)^5}$$

13. Solve
$$\lim \frac{\sqrt{n}(2n-4)}{\sqrt{n^3+3n}}$$

14. Solve
$$\lim \frac{(2n^2-4n+1)\sqrt{n}}{\sqrt{n^5+4n}}$$

15. Solve
$$\lim \sqrt[3]{n}^2(\sqrt[3]{n} - \sqrt[3]{n-1})$$

16. Solve
$$\lim \frac{3^n + n2^n}{4^n}$$

17. Solve
$$\lim \frac{e^{5n}}{3 - e^{2n}}$$

18. Solve
$$\lim \frac{n^2+1}{(-1-n)(n+2)}$$

19. Solve
$$\lim \sqrt{n}(\sqrt{2n} - \sqrt{2n-1})$$

20. Solve
$$\lim \frac{1-n^3}{n+3}$$

21. Solve
$$\lim \frac{\log(n+2)}{\log(1+4n)}$$

2.3 series – introduction

1. Find values of
$$\sum_{n=0}^{5} 2^n,\quad \sum_{n=0}^{8} 2^n,\quad \sum_{n=1}^{5} 2^n.$$

2. Write the first three partial sums of
$$\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+1}.$$

3. Make an index shift so that
$$\sum_{n=7}^{\infty} \frac{4-n}{n^2+1}.$$
starts at $n = 3$.

4. Write the first three partial sums of
$$\sum_{n=3}^{\infty} \frac{2n}{n+2}.$$

5. Evaluate
$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n.$$

6. Find the value of $a \in \mathbb{R}$ such that
$$\sum_{n=0}^{\infty} \frac{ax^n}{n!} = 1.$$
Find also the value of $b \in \mathbb{R}$ such that
$$\sum_{n=2}^{\infty} \frac{bx^n}{n!} = 1.$$
(Hint: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

2.4 series – positive summands

1. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{\sin^2 n \left(2n^3+7\right)}{n^5+1}$$

2. Examine
$$\sum_{n=1}^{\infty} \frac{7}{n(n+1)}.$$

3. Decide about the convergence
$$\sum_{n=1}^{\infty} n\frac{2^n}{3^n}.$$

4. Examine
$$\sum_{n=1}^{\infty} \frac{n^2+9n}{4+2n^2}.$$

5. Examine
$$\sum_{n=2}^{\infty} \frac{n^2}{n^3-3}.$$

6. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{2^{3n}(n+1)}{n^2 5^{1+n}}.$$

7. Examine the convergence of
$$\sum_{n=4}^{\infty} \frac{5^{2n}}{2^{5n-3}}.$$

8. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{[(n+1)!]^n}{2! \cdot 4! \cdots (2n)!}$$

9. Examine
$$\sum_{n=1}^{\infty} \frac{5}{6n}.$$

10. Examine
$$\sum_{n=3}^{\infty} \frac{1}{\sqrt{2n-3}\sqrt{2n+3}}.$$

11. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{6^n}.$$

12. Examine
$$\sum_{n=1}^{\infty} \left(e^{\frac{1}{n^2}}-1\right).$$

13. Examine
$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 5^n}.$$

14. Examine
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+3}.$$

15. Examine
$$\sum_{n=1}^{\infty} \left(\frac{1+\cos n}{2+\cos n}\right)^n.$$

16. Decide about the convergence of
$$\sum_{n=1}^{\infty} \frac{n^{n+1/n}}{\left(n+\frac{1}{n}\right)^n}.$$

2.5 series – general case

1. Examine
$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}.$$

2. Decide about the convergence of
$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) n^{-2}.$$

3. Examine
$$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+1} \frac{1}{\sqrt[100]{n}}.$$

4. Decide about the convergence of
$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}+2}{n}.$$

5. Examine
$$\sum_{n=1}^{\infty}(-1)^{n(n-1)}\frac{1}{n}.$$

6. Examine
$$\sum_{n=1}^{\infty}(-1)^{n}\frac{2+(-1)^n}{n}$$

3 Functions of multiple variables

3.1 Intro – topology

1. Find the boundary of
$$M=\{(x,y)\in\mathbb{R}^2,\, x>y,\, y>x^2\}.$$

2. Determine whether
$$M=\{(x,y)\in\mathbb{R}^2,\ x^2>y,\ y>1\}$$
is open or closed.

3. For every set given below, give one interior and one boundary point and determine whether the set is open or closed

   (a) $A=\{(x,y)\in\mathbb{R}^2,\, \frac{1}{4}(x-1)^2+\frac{1}{9}(y-3)^2\le 1\}$
   (b) $B=\{(x,y)\in\mathbb{R}^2,\, y\neq x^2\}$
   (c) $C=\{(x,y)\in\mathbb{R}^2,\, x^2+y^2=1\}$
   (d) $D=\{(x,y)\in\mathbb{R}^2,\, y>\sin x\}$

3.2 Intro – functions

1. Determine and sketch domains of the following functions
$$a(x)=\frac{1}{\ln(x^2-y)},\quad b(x)=\sqrt{\frac{x+y}{x-y}}$$
$$c(x)=\sqrt{2-|x+y|},\quad d(x)=\sqrt{x^2+xy+1}$$
$$e(x)=\sqrt{\frac{1-|x|}{|y|-1}},\quad f(x)=x-\frac{3}{y}+\ln(x-3y).$$

2. Determine and sketch contour lines at heights $-2,-1,0,1,2$ of
$$a(x)=x^2+y,\quad b(x)=x^2-y^2$$
$$c(x)=|x|+|2-y|,\quad d(x)=\frac{1}{1+x^2+y^2}$$
$$e(x)=\frac{x+1}{y-2},\quad f(x)=\frac{x^2+y^2}{2x+y}.$$

3. Let
$$f(x,y)=\frac{x^2+y}{x+y^2}.$$
Find its cross-section along the line
$$p:(x,y)=(1,1)+t(-2,1),\ t\in\mathbb{R}.$$

4. Write $g(t)=f(1+t,t^2)$ where
$$f(x,y)=x^2+\sqrt{y},$$
determine its domain and sketch its graph.

5. Examine
$$\lim_{(x,y)\to(0,0)}\frac{2xy}{3x^2+y^2},\quad \lim_{(x,y)\to(0,0)}\frac{4xy^2}{x^2+3y^4}$$
$$\lim_{(x,y)\to(2,1)}\frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2},\quad \lim_{(x,y)\to(0,0)}\frac{xy}{x^3+y^3}$$
$$\lim_{(x,y)\to(a,a)}\frac{x^4-y^4}{x^3-y^3},\ a\in\mathbb{R},\quad \lim_{(x,y)\to(0,0)}\frac{\sin(xy)}{x+y}$$
$$\lim_{(x,y)\to(0,0)}\frac{5x^2y^2}{x^2+y^2},\quad \lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$$

6. Compute
$$\lim_{x\to0}\left(\lim_{y\to0}\frac{xy+\sin(xy)}{xy}\right),\quad \lim_{y\to0}\left(\lim_{x\to0}\frac{(x^2-2xy+y^2)}{xy}\right)$$

3.3 Derivatives

- Compute the derivative of f with respect to direction v at point (x_0,y_0) where

  1. $f(x,y)=x^2+2y$, $v=\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$, $(x_0,y_0)=(-1,0)$,
  2. $f(x,y)=y^2\sin x$, $v=\left(\frac{3}{5},\frac{4}{5}\right)$, $(x_0,y_0)=(2,5)$,
  3. $f(x,y)=x+e^{xy}$, $v=\left(\frac{12}{13},\frac{5}{13}\right)$, $(x_0,y_0)=(0,0)$.

- Compute first partial derivatives of the following functions
$$a(x,y)=e^{x(y^2+xy)},\quad b(x,y)=(x^2+y)\sin x$$
$$c(x,y,z)=\frac{xe^y}{z+x},\quad d(x,y)=(x^2+3xy)^{\ln(xy)}$$
$$e(x,y)=(x^2+2y)\cos(xy),\quad f(x,y)=\frac{(2x-3y)^5}{x^2-1}$$

- Compute ∇^2 of the following functions
$$a(x,y)=x^y,\quad b(x,y)=\frac{x^2}{y}$$
$$c(x,y,z)=\frac{x+y}{y+z},\quad d(x,y)=\frac{\sqrt{x}}{y^2+1}$$
$$e(x,y)=x\sqrt{x^2+y^2},\quad f(x,y)=(x^2+y^2)^{-2}(x-y)^3$$

- Write the tangent plane to the graph of function
$$f(x,y)=\frac{1}{\sqrt{x^2+y}}$$
at point $(x_0,y_0)=(2,5)$.

- Write the tangent plane to the graph of function
$$f(x,y)=\frac{\ln(x+3y)}{(x-1)y}$$

at point $(x_0, y_0) = (-2, 1)$.

- Let $f(x,y) = x\sin(x+y^2)$. Let $x(t) = t^2 + t$ and $y(t) = e^t \ln t$. Compute
$$\frac{\partial}{\partial t} f(x(t), y(t)).$$
- Write the second order Taylor polynomial at point $(1,1)$ for
$$f(x,y) = \ln(2y - x^2).$$
- Use the second order Taylor polynomial to deduce an approximate value of
$$2^{(1.9)^2+(0.12)^2}.$$
- Write the second order Taylor polynomial at point $(-1,1,2)$ for
$$f(x,y,z) = x^2 + (2xy)(z+3).$$
- Use the second order Taylor polynomial to deduce an approximate value of
$$\sqrt{(2.1)^2 + (1.9)^2 + (1.1)^2}.$$

3.4 Implicit function theorem

1. Show that the equation
$$2x^2 + e^y = 3$$
uniquely determines a function $y(x)$ in the vicinity of $(1,0)$. Compute $y'(1)$ and $y''(1)$.
2. Use the implicit function theorem to show that the equation
$$xy^2 - \sin y = 0$$
determines a function $y(x)$ in the neighborhood of $(3,0)$. Compute $y'(3)$.
3. Find a tangent line to a curve
$$e^{xy} + \sin y + y^2 = 1$$
passing through $(2,0)$.
4. Is there a function $y(x)$ determined by
$$y + xy^2 - xe^x = 0$$
in the neighborhood of $(0,0)$? If yes, write its approximation by the second order Taylor polynomial.

3.5 Extremes

1. Find local extremes of a function
$$f(x,y) = x^3 + 8y^3 - 6xy + 5.$$
2. Find all local maxima and minima of $f(x,y) = e^{2x+3y}(8x^2 - 6xy + 3y^2)$.
3. Find all local maxima and minima of a function
$$f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$$
4. Find the global maximum and minimum values of $f(x,y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.
5. Find the global maximum and minimum values of $f(x,y) = x^2 + 2y^2 - 4y$ subject to the constraint $x^2 + y^2 = 9$.
6. Find the global maximum and minimum values of $f(x,y) = x^2 + xy$ subject to the constraint $y \le 9$, $y \ge x^2$.
7. Find the maximum and minimum values of $f(x,y) = x^2 + y^2$ on the set $M = \{(x,y) \in \mathbb{R}^2,\ xy \le 1,\ x \ge 1/2,\ y \ge 1/2\}$.
8. Find the maximum and minimum of $f(x,y,z) = xyz$ on a set $M = \{x^2 + y^2 + z^2 = 1,\ x + y + z = 0\}$.
9. Find the maximum and minimum of $f(x,y,z) = z + e^{xy}$ on a set $M = \{x^2 + y^2 + z^2 = 1,\ x^2 + y^2 = z\}$.
10. Find the maximum and minimum of $f(x,y,z) = x^2 + 2xz + y^2 + z$ on a set $M = \{x^2 + y^2 + z^2 \le 1,\ x = y^2 + z^2\}$.

4 Systems of differential equations

4.1 Introduction

1. Express
$$x'' - 3x' + y - x = 0$$
$$y''' + y'' - x' + y' + x = 0$$
as a system of first-order differential equations.
2. Rewrite the system
$$x' = 2x + 5 - 3y$$
$$y' = 6 - z - x - y$$
$$z' = x + z + 20$$
into a matrix form (i.e., $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ where $\mathbf{x}$ is the vector whose components are x, y and z).

4.2 Linear systems

1. Find all solutions to
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
2. Find the fundamental system set for $x' = Ax$ where
   (a)
$$A = \begin{pmatrix} -1 & 1 \\ 8 & 1 \end{pmatrix},$$
   (b)
$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix}.$$
3. Find the general solution to the system
$$\begin{aligned} x' &= x + 2y - z \\ y' &= x + z \\ z' &= 4x - 4y + 5z. \end{aligned}$$
4. Find all solutions to
$$x' = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} x$$
5. Solve the initial value problem
$$\begin{aligned} x' &= 4x + y \\ y' &= -2x + y \\ (x(0), y(0)) &= (1, 0) \end{aligned}$$
6. Find the solution to
$$x' = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} x$$
   which satisfies
$$x(0) = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$$
7. Solve the initial value problem
$$x' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
8. Find all solution to
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -11 \\ -5 \end{pmatrix}$$
9. Solve the initial value problem
$$\begin{aligned} x' - 2y = 4t, &\quad x(0) = 4 \\ y' + 2y - 4x = -4t - 2, &\quad (0) = -5. \end{aligned}$$

4.3 Phase plane

1. Verify, that the $x(t) = e^{3t}$ and $y(t) = e^t$ is a solution to
$$\begin{aligned} x' &= 3y^3 \\ y' &= y \end{aligned}$$
   and sketch this particular trajetory into the phase plane.
2. Find the critical points of
$$\begin{aligned} x' &= y^2 - 3y + 2 \\ y' &= (x-1)(y-2) \end{aligned}$$
3. Find all the critical points of the system
$$\begin{aligned} x' &= x^2 - 1 \\ y' &= xy \end{aligned}$$
   and sketch few representative trajectories into the phase plane.
4. Find the critical points and solve the related phase plane diagram equation for the system
$$\begin{aligned} x' &= (x-1)(y-1) \\ y' &= y(y-1). \end{aligned}$$

4.4 Stability of critical points

1. Classify the critical point at the origin of
$$\begin{aligned} x' &= 5x + 6y \\ y' &= -5x - 8y. \end{aligned}$$
2. Classify the critical point at the origin of
$$\begin{aligned} x' &= 6x - y \\ y' &= 8y. \end{aligned}$$
3. Find and classify the critical point of
$$\begin{aligned} x' &= -4x + 2y + 8 \\ y' &= x - 2y + 1. \end{aligned}$$
4. Find and classify the critical point of
$$\begin{aligned} x' &= 2x + 4y + 4 \\ y' &= 3x + 5y + 4. \end{aligned}$$
5. A simplified model for an arms race between two countries whose expenditures for defense are expressed by the variables $x(t)$ and $y(t)$ is given by linear system
$$\begin{aligned} x' &= 2y - x + a, \quad x(0) = 1, \\ y' &= 4x - 3y + b, \quad y(0) = 4. \end{aligned}$$
   where a and b are constants that measure the trust (or distrust) each country has for the other. Determine whether there is going to be disarment (x and y approach 0 as t increases), a stabilized arms race (x and y approach a constant as $t \to \infty$), or a runaway arms race (x and y approach ∞ as $t \to \infty$).
6. Show, that the system
$$\begin{aligned} x' &= 3x + 2y - y^2 \\ y' &= -2x - 2y + xy \end{aligned}$$
   is almost linear near the origin.

7. Find the critical points of

$$x' = x + y$$
$$y' = 5y - xy + 6$$

and discuss their type and stability.

8. Find the critical points of

$$x' = 4 - xy$$
$$y' = x - y$$

and discuss their type and stability.