1 Introduction

1.1 Mathematical induction

1. Show that
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$
holds for every $n\in\mathbb{N}$

2. Show that
$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$$
holds for every $n\in\mathbb{N}$.

3. Show that
$$2^n>n^2$$
for every $n\in\mathbb{N}$.

4. Show that 6 is a divisor of 10^n-4 for every natural number n.

2 Linear algebra

2.1 Vectors

1. Let $u=(-1,5)$, $v=(2.7,3.8)$, $w=(4.2,-6)$. Compute $u+v$, $u-v$, $v-u$, $v-w$, $u+2w$, $3u-v$, $u+v+w$, $3u-2v-5w$, $\frac{1}{2}w+\frac{3}{4}v-\frac{5}{2}u$.

2. Are vectors
$$v_1=(1,1,1,2)$$
$$v_2=(1,2,-1,1)$$
$$v_3=(0,1,1,2)$$
linearly independent?

3. Check if the vector $u=(2,0,-1)$ is a linear combination of $v=(6,-2,4)$ and $w=(-3,-1,2)$.

4. Find the value of $k\in\mathbb{R}$ such that the vector $u=(k,4,k)$ is a linear combination of $v=(-1,2,2)$ and $w=(4,2,1)$.

5. Polynomial $P(x)=x^2+3x+c$, $c\in\mathbb{R}$ belongs to a linear span of $Q(x)=2x^2-1$ and $R(x)=x+2$. Determine the value of c.

6. Find λ_1, λ_2 and λ_3 such that
$$\lambda_1(2,1,3)+\lambda_2(0,-2,0)+\lambda_3(1,4,2)=(9,9,14)$$
(remark: λ_1, λ_2, λ_3 are coordinates of $(9,9,14)$ with respect to the basis
$$\{(2,1,3),\ (0,-2,0),\ (1,4,2)\}.)$$

7. Let
$$u=(2,4,6),\ v=(-1,-2,-3),\ w=(-2,-4,-6).$$
Is there a vector z such that $u\notin\operatorname{span}\{v,w,z\}$?

8. How the dimension of
$$\operatorname{span}\{(1,1,1),(2,2,2),(3,3,k)\}$$
depends on k?

9. Determine, whether b is a linear combination of a_1, a_2 and a_3 where
$$a_1=(1,-2,2),\ a_2=(0,5,5),$$
$$a_3=(2,0,8),\ b=(-5,11,8).$$

10. Let
$$M_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ M_2=\begin{pmatrix}0&0\\0&1\end{pmatrix},\ M_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$
Show that $\operatorname{span}\{M_1,M_2,M_3\}$ is a space of all symmetric matrices.

2.2 Matrices – intro

1. Let
$$A=\begin{pmatrix}1&1\\-2&1\end{pmatrix},\ B=\begin{pmatrix}1&2\end{pmatrix},\ C=\begin{pmatrix}-1\\1\end{pmatrix}.$$
Compute (if possible)
$$2A+BC,\ AB+C,\ CB-A,\ 2BA+C,\ B\left(C+\begin{pmatrix}2\\-1\end{pmatrix}\right)$$

2. Is there $\alpha\in\mathbb{R}$ such that the matrix
$$A=\begin{pmatrix}1&\alpha&1\\0&1&0\\1&0&1\end{pmatrix}$$
has a rank 3?

3. Do the vectors
$$(0,0,-2),\ (0,-3,8),\ (4,-1,-5)$$
span $\mathbb{R}^3$?

4. Determine the rank of the matrix $A=\begin{pmatrix}3&2&3\\-2&2&1\\3&0&1\end{pmatrix}$ by using the definition of rank.

2.3 Matrices – the Gauss elimination method

1. Determine the rank of

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 2 & 4 & 5 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 2 & 5 & -1 \end{pmatrix}$$

2. Solve

$$\begin{aligned} 3x + y - z &= 1 \\ x - y + z &= -3 \\ 2x + y + z &= 0 \end{aligned}$$

3. Solve

$$\begin{aligned} 2x + 5y &= 9 \\ x + 2y - z &= 3 \\ -3x - 4z + 7z &= 1 \end{aligned}$$

4. An amount of \$65,000 is invested in three bonds at the rates of 6%, 8% and 10% per annum respectively. The total annual income is \$4,800. The income from the third bond is \$600 more than that from the second bond. Determine the price of each bond.

5. Find all solutions to

$$\begin{aligned} 2x - 3y + z &= 2 \\ -x + 2y - z &= -2 \\ 3x - 4y + z &= 2 \end{aligned}$$

2.4 Square matrices – intro

1. Verify that

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}.$$

2. Determine

$$\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}^{-1}$$

3. Compute

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}^{-1}$$

4. Is the matrix X solving

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} X + \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

regular or singular?

2.5 Square matrices – determinants

1. Compute

$$\det \begin{pmatrix} \lambda - 2 & 3 \\ 1 - \lambda & 2 \end{pmatrix}$$

where $\lambda \in \mathbb{R}$.

2. Compute

$$\det \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

3. Compute

$$\det \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 3 & 3 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

4. Use the Cramer rule to solve

$$\begin{aligned} 2x - y + z &= 3 \\ 3x + 2y + z &= 7 \\ -x - y - 2z &= -6 \end{aligned}$$

5. Compute

$$\det \begin{pmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 & 3 \\ -2 & 0 & 0 & 0 & 3 \\ 1 & -1 & 1 & -1 & 3 \end{pmatrix}$$

6. Find $\alpha \in \mathbb{R}$ for which is the matrix

) (

singular

2.6 Square matrices – eigenvalues

1. Determine the eigenvalues and eigenvectors (including the generalized ones) of a matrix

$$A = \begin{pmatrix} -5 & -3 & -1 \\ 16 & 9 & 3 \\ -2 & -1 & 1 \end{pmatrix}$$

2. Find all eigenvectors and eigenvalues of

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

3. Find a matrix whose characteristic polynomial is $-\lambda^3 + 2\lambda^2 + \lambda - 2$.

4. Find eigenvalues (not eigenvectors) of

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 4 & 3 & -2 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 2 & 3 \end{pmatrix}.$$

5. Find all vectors (including the generalized) of
$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -4 & 4 \end{pmatrix}.$$

2.7 Square matrices – quadratic forms and definiteness

1. Write the quadratic form Q whose corresponding matrix is
   - $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
   - $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix}$
2. Write the symmetric matrix corresponding to
   - $Q(x, y) = x^2 + 3xy - y^2$
   - $Q(x, y, z) = 2x^2 - 5xy + 3yz - 2z^2$
   - $Q(x, y, z) = x(x + 2y) + (3x - y)(2x - 2y)$
3. Decide about the definiteness of the following matrices:
   - $\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$
   - $\begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 2 \end{pmatrix}$
   - $\begin{pmatrix} -2 & -2 & -1 & 1 \\ -2 & -4 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}$

3 Functions

3.1 Domains

1. Find the maximal domain of $f(x) = \sqrt{x^2 - 5x + 6}$.
2. Find the maximal domain of $f(x) = \frac{x}{x^2-4}$.
3. Find the maximal domain of $f(x) = \frac{1}{\sqrt{16-x^2}}$.
4. Find the maximal domain of $f(x) = \log(2x - x^2)$.
5. Find the maximal domain of $f(x) = \sqrt{3 - \log x}$.
6. Find the maximal domain of $f(x) = \sqrt[4]{x \log(x + 2)}$.

3.2 Parity

1. Decide about the parity of $f(x) = x \sin x$.
2. Decide about the parity of $f(x) = x^3 + 2$.
3. Decide about the parity of $f(x) = xe^x$.
4. Decide about the parity of $f(x) = \log \frac{2-x}{2+x}$.

3.3 Inverse function

1. Find f^{-1} (if possible) for $f(x) = \frac{1}{\sqrt{9+x^2}}$.
2. Find f^{-1} (if possible) for $f(x) = 1 - \sqrt{x + 2}$.
3. Find f^{-1} (if possible) for $f(x) = \frac{\sqrt{x^2-4}}{x}$, $x \in [2, \infty)$.
4. Find f^{-1} (if possible) for $f(x) = 3 - \frac{5}{x+1}$.
5. Find f^{-1} (if possible) for $f(x) = 3^{\frac{x+1}{x+3}}$.

3.4 Limits 1

1. Solve
$$\lim_{x\to 3} x + 2$$
2. Solve
$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x^2 + 3x - 18}$$
3. Solve
$$\lim_{x\to +\infty} \frac{1}{2}x^3 - x + 1$$
4. Solve
$$\lim_{x\to +\infty} \frac{x + 2}{x^2 + 3}$$
5. Solve
$$\lim_{x\to -\infty} \frac{x^4}{x + 1}$$
6. Solve
$$\lim_{x\to 0} \frac{x^3 + x + 1}{x^2 + x}$$
7. Solve
$$\lim_{x\to -1} \frac{x^3 - x + 2}{x^2 + 2x + 1}$$
8. Solve
$$\lim_{x\to +\infty} \frac{x^3 + 2x}{x^3 - 1}$$
9. Solve
$$\lim_{x\to -\infty} \sqrt{x^2 + 1}$$
10. Solve
$$\lim_{x\to 4} \frac{x^2 - 18}{x^2 - 8x + 16}$$

11. Solve
$$\lim_{x\to 0}\frac{\sin 3x}{x}$$
12. Solve
$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}$$
13. Solve
$$\lim_{x\to\infty} e^{-x}$$
14. Solve
$$\lim_{x\to -\infty}\frac{\sin x}{x}$$
15. Solve
$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$$
16. Solve
$$\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{e^{2x}-e^{-2x}}$$
17. Solve
$$\lim_{x\to 0+} x^x$$
18. Solve
$$\lim_{x\to 1}\frac{\log x}{e^{x-1}-1}$$

3.5 Derivatives

1. $$\left(x^2+\frac{1}{\sqrt{x}}\right)'$$
2. $$\left(\frac{x+1}{x^2+1}\right)'$$
3. $$\left(\sqrt{x^2+5}\right)'$$
4. $$\left(e^{x+x^2\sin x}\right)'$$
5. $$\left(\frac{x^3+x\sin x}{\sqrt{x^2+1}}\right)'$$
6. $$\left(\sin\left(\frac{x^2+1}{x}\right)\right)'$$
7. $$\left(\sqrt{\sin(x^2)}\right)'$$
8. $$\left(\sqrt{1-\sin^2 x}\right)'$$
9. $$\left(\sqrt{\sqrt{\sqrt{1+x^2}}}\right)'$$
10. $$\left(\frac{2x}{x^2+4}\right)''$$
11. $$\left(x\sqrt[3]{1+\sin^2 x}\right)''$$
12. $$\left(e^{2x^2+x}\right)'''$$

3.6 Tangent lines

1. Find the tangent line to the graph of $f(x)=e^{x^2-1}$ at $A=[-1,1]$.
2. Find the tangent line to the graph of $f(x)=1+\frac{1}{x}$ at $A=[1,?]$.
3. Find the tangent line to the graph of $f(x)=3-x^2$ whose slope is $k=-2$.
4. Find the tangent line to the graph of $f(x)=\frac{x^2}{x+1}$ which is parallel to $y=4-x$.

3.7 The course of f

1. Draw the graph of the function f with the following properties: $\operatorname{Dom} f=\mathbb{R}$, even, does not have a derivative at $x=0$, $f'>0$ on $(-\infty,-1)$, $f(0)=-2$.
2. Draw the graph of the function f with the following properties: $\operatorname{Dom} f=(0,\infty)$, $f(5)=0$, $f'(x)<4$ on $(0,4)$ and $f'(x)>0$ on $(4,\infty)$, $\lim_{x\to 0+}=5$.
3. Examine the course of
$$f(x)=x^3-5x^2+3x-5.$$
4. Examine the course of
$$f(x)=x^2(4-x)^2.$$
5. Examine the course of
$$f(x)=\frac{x^2+1}{x^2-1}.$$
6. Examine the course of
$$f(x)=(1+\cos x)\sin x.$$

4 Functions of multiple variables

4.1 Introduction

1. Determine and sketch domains of the following functions

$$a(x) = \frac{1}{\ln(x^2 - y)}, \quad b(x) = \sqrt{\frac{x+y}{x-y}}$$
$$c(x) = \sqrt{2 - |x+y|}, \quad d(x) = \sqrt{x^2 + xy + 1}$$
$$e(x) = \sqrt{\frac{1-|x|}{|y|-1}}, \quad f(x) = x - \frac{3}{y} + \ln(x - 3y).$$

2. Determine and sketch contour lines at heights -2, -1, 0, 1, 2 of

$$a(x) = x^2 + y, \quad b(x) = x^2 - y^2$$
$$c(x) = |x| + |2 - y|, \quad d(x) = \frac{1}{1 + x^2 + y^2}$$
$$e(x) = \frac{x+1}{y-2}, \quad f(x) = \frac{x^2 + y^2}{2x + y}.$$

3. Let

$$f(x,y) = \frac{x^2 + y}{x + y^2}.$$

Find its cross-section along the line

$$p : (x,y) = (1,1) + t(-2,1), \ t \in \mathbb{R}.$$

4. Write $g(t) = f(1 + t, t^2)$ where

$$f(x,y) = x^2 + \sqrt{y},$$

determine its domain and sketch its graph.

4.2 Topology

1. Find the boundary of

$$M = \{(x,y) \in \mathbb{R}^2,\, x > y,\, y > x^2\}.$$

2. Determine whether

$$M = \{(x,y) \in \mathbb{R}^2, \ x^2 > y, \ y > 1\}$$

is open or closed.

3. For every set given below, give one interior and one boundary point and determine whether the set is open or closed

(a) $A = \{(x,y) \in \mathbb{R}^2,\, \frac{1}{4}(x-1)^2 + \frac{1}{9}(y-3)^2 \leq 1\}$

(b) $B = \{(x,y) \in \mathbb{R}^2,\, y \neq x^2\}$

(c) $C = \{(x,y) \in \mathbb{R}^2,\, x^2 + y^2 = 1\}$

(d) $D = \{(x,y) \in \mathbb{R}^2,\, y > \sin x\}$

4.3 Limit and continuity

1. Examine

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2 + y^2}, \quad \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4}$$
$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2 + (y-1)^2}, \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^3 + y^3}$$
$$\lim_{(x,y)\to(a,a)} \frac{x^4 - y^4}{x^3 - y^3}, \ a \in \mathbb{R}, \quad \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x + y}$$
$$\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2 + y^2}, \quad \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

2. Compute

$$\lim_{x\to 0} \left(\lim_{y\to 0} \frac{xy + \sin(xy)}{xy} \right), \quad \lim_{y\to 0} \left(\lim_{x\to 0} \frac{(x^2 - 2xy + y^2)}{xy} \right)$$

4.4 Derivatives

- Compute the derivative of f with respect to direction v at point (x_0, y_0) where

1. $f(x,y) = x^2 + 2y$, $v = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $(x_0, y_0) = (-1, 0)$,
2. $f(x,y) = y^2 \sin x$, $v = \left(\frac{3}{5}, \frac{4}{5}\right)$, $(x_0, y_0) = (2, 5)$,
3. $f(x,y) = x + e^{xy}$, $v = \left(\frac{12}{13}, \frac{5}{13}\right)$, $(x_0, y_0) = (0, 0)$.

- Compute first partial derivatives of the following functions

$$a(x,y) = e^{x(y^2 + xy)}, \quad b(x,y) = (x^2 + y)\sin x$$
$$c(x,y,z) = \frac{xe^y}{z + x}, \quad d(x,y) = (x^2 + 3xy)^{\ln(xy)}$$
$$e(x,y) = (x^2 + 2y)\cos(xy), \quad f(x,y) = \frac{(2x - 3y)^5}{x^2 - 1}$$

- Compute ∇^2 of the following functions

$$a(x,y) = x^y, \quad b(x,y) = \frac{x^2}{y}$$
$$c(x,y,z) = \frac{x+y}{y+z}, \quad d(x,y) = \frac{\sqrt{x}}{y^2 + 1}$$
$$e(x,y) = x\sqrt{x^2 + y^2}, \quad f(x,y) = (x^2 + y^2)^{-2}(x - y)^3$$

- Write the tangent plane to the graph of function

$$f(x,y) = \frac{1}{\sqrt{x^2 + y}}$$

at point $(x_0, y_0) = (2, 5)$.

- Write the tangent plane to the graph of function

$$f(x,y) = \frac{\ln(x + 3y)}{(x-1)y}$$

at point $(x_0, y_0) = (-2, 1)$.

- Let $f(x,y) = x\sin(x+y^2)$. Let $x(t) = t^2 + t$ and $y(t) = e^t \ln t$. Compute
$$\frac{\partial}{\partial t} f(x(t), y(t)).$$
- Write the second order Taylor polynomial at point $(1,1)$ for
$$f(x,y) = \ln(2y - x^2).$$
- Use the second order Taylor polynomial to deduce an approximate value of
$$2^{(1.9)^2+(0.12)^2}.$$
- Write the second order Taylor polynomial at point $(-1,1,2)$ for
$$f(x,y,z) = x^2 + (2xy)(z+3).$$
- Use the second order Taylor polynomial to deduce an approximate value of
$$\sqrt{(2.1)^2 + (1.9)^2 + (1.1)^2}.$$