Integrals

Václav Mácha

University of Chemistry and Technology

Examples

- Recall that total revenue and marginal revenue are related as

$$MR = \frac{\partial TR}{\partial Q}.$$

  Find the total revenue function if

$$MR = 100 + 20Q + 3Q^2$$

  and $TR(2) = 260$.

- The train begins its motion with a velocity v (in meters per seconds) given as

$$v(t) = \frac{20t}{t+20}$$

  Determine the distance the train has traveled after 30 seconds.

Definition

We say that F is a *primitive function* (on interval (a, b)) of f if $F'(x) = f(x)$ (for all $x \in (a, b)$).

We will use also the following notation

$$\int f(x)\,\mathrm{d}x = F(x).$$

Observation

Let F_1 and F_2 be two primitive functions of f on interval (a, b). Then $F_1 - F_2 \equiv c$ for some constant $c \in \mathbb{R}$.

Proof: It suffices to consider $(F_1 - F_2)' = (f - f) = 0$. The claim follows immediately.

As a consequence the primitive function is not determined uniquely. In particular, the primitive function to a given function f is a whole set of functions which differ by arbitrary constant – if F is a primitive function of f then all functions in form $F + c$, $c \in \mathbb{R}$ are also primitive functions.

Observation

Let F be a primitive function of f and G be a primitive function of g. Then $F + G$ is a primitive function of $f + g$ and cF is a primitive function of cf for every $c \in \mathbb{R}$.

Proof: It is enough to use rules for derivatives.

Further, we may use the table of basic derivatives in an 'inverted' way:

$f(x)$	$F(x)$	conditions
x^n	$\frac{1}{n+1}x^{n+1} + c,\ c \in \mathbb{R}$	$n \neq -1$, x as usual
x^{-1}	$\log\|x\| + c,\ c \in \mathbb{R}$	$x \neq 0$
e^x	$e^x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
a^x	$\frac{1}{\log a}a^x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R},\ a \in (0,1) \cup (1,\infty)$
$\sin x$	$-\cos x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
$\cos x$	$\sin x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
$\frac{1}{1+x^2}$	$\arctan x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c,\ c \in \mathbb{R}$	$x \in (-1,1)$

Exercises:

- $$\int 2x^2 - 4x + 12\,\mathrm{d}x$$
- $$\int \frac{x+1}{\sqrt{x}}\,\mathrm{d}x$$
- $$\int \frac{x^2}{x^2+1}\,\mathrm{d}x$$
- $$\int \frac{2^{x+1} - 5^{x-1}}{10^x}\,\mathrm{d}x$$
- $$\int \left(1 - \frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,\mathrm{d}x$$

Linear substitution

Observation

Let $F(x)$ be a primitive function to $f(x)$. Then $\frac{1}{a}F(ax+b)$ is a primitive function of $f(ax+b)$.

Proof: Indeed, we derive the composed function $F(ax+b)$:

$$(F(ax+b))' = F'(ax+b)(ax+b)' = f(ax+b)a.$$

Exercises:

- $$\int (2x+3)^7 \,\mathrm{d}x$$
- $$\int \frac{1}{x^2+4} \,\mathrm{d}x$$
- $$\int \frac{e^{3x}+1}{e^x+1} \,\mathrm{d}x$$
- $$\int \sqrt[3]{1-3x} \,\mathrm{d}x$$
- $$\int \frac{1}{1-x} + \frac{1}{1+x} \,\mathrm{d}x$$

The method of substitution

Theorem

Let $\varphi : (\alpha, \beta) \to (a, b)$ has a finite derivative in every $x \in (\alpha, \beta)$ and let f be defined on (a, b). Then

$$\int f(x)\,\mathrm{d}x = \int f(\varphi(x))\varphi'(x)\,\mathrm{d}x$$

Example: Solve $\int \sin^2 x \cos x \,\mathrm{d}x$:

The method of substitution

Theorem

Let $\varphi : (\alpha, \beta) \to (a, b)$ has a finite derivative in every $x \in (\alpha, \beta)$ and let f be defined on (a, b). Then

$$\int f(x)\,\mathrm{d}x = \int f(\varphi(x))\varphi'(x)\,\mathrm{d}x$$

Example: Solve $\int \sin^2 x \cos x \,\mathrm{d}x$:

$$\int \sin^2 x \cos x \,\mathrm{d}x = \begin{bmatrix} \sin x = t \\ \cos x \,\mathrm{d}x = 1\,\mathrm{d}t \end{bmatrix} =$$

$$\int t^2 \,\mathrm{d}t = \frac{t^3}{3} + c = \frac{\sin^3 x}{3} + c,\ c \in \mathbb{R}.$$

Exercises

- $$\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$$
- $$\int \frac{\log^4 x}{x}\,\mathrm{d}x$$
- $$\int 3e^x\sqrt{1+e^x}\,\mathrm{d}x$$
- $$\int e^{\sin x}\cos x\,\mathrm{d}x$$
- $$\int x^5\sqrt{x^3+8}\,\mathrm{d}x$$

The integration by parts

Theorem

Let F be the primitive function of f and G be the primitive function of g. Then

$$\int F(x)g(x)\,\mathrm{d}x = F(x)G(x) - \int f(x)G(x)\,\mathrm{d}x$$

Integrals of the type $\int Polynomial(e^x, \sin x, \cos x)\,\mathrm{d}x$:

Exercises

- $$\int xe^{3x}\,\mathrm{d}x$$
- $$\int (x^2 - 2x)\cos x\,\mathrm{d}x$$

Integrals of the type $\int(e^x, \sin x, \cos x)(e^x, \sin x, \cos x)\,\mathrm{d}x$:

Exercises

- $$\int e^x \sin x \,\mathrm{d}x$$
- $$\int \sin 2x \cos 4x \,\mathrm{d}x$$
- $$\int \cos^2 x \,\mathrm{d}x$$

Integrals of the type $\int \mathit{Polynomial}(\log x, \arctan x)\,\mathrm{d}x$:

Exercises

- $$\int x^3 \log x \,\mathrm{d}x$$
- $$\int \arctan x \,\mathrm{d}x$$

Both methods at once: Exercises

- $$\int \cos^5 x \sqrt{\sin x}\, \mathrm{d}x$$
- $$\int x^5 e^{x^3}\, \mathrm{d}x$$
- $$\int \sin(\log x)\, \mathrm{d}x$$
- $$\int \frac{\sin x \cos^3 x}{1+\cos^2 x}\, \mathrm{d}x$$

Rational functions

Functions of the form $\frac{R(x)}{P(x)}$ where R and P are polynomials.

Examples

- Solve $\int \frac{6}{4-x}\,\mathrm{d}x$
- Solve $\int \frac{x^3-x}{2x+3}\,\mathrm{d}x$
- Solve $\int \frac{x^3+2x+1}{x^2+2}\,\mathrm{d}x$

Rational functions

Functions of the form $\frac{R(x)}{P(x)}$ where R and P are polynomials.

Examples

- Solve $\int \frac{6}{4-x}\,\mathrm{d}x$
- Solve $\int \frac{x^3-x}{2x+3}\,\mathrm{d}x$
- Solve $\int \frac{x^3+2x+1}{x^2+2}\,\mathrm{d}x$

Lemma

The integral of $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$ *is* $\log|f(x)|$.

Examples

- Solve $\int \frac{x^3+4x}{x^2+2}\,\mathrm{d}x$.

Theorem

Every polynomial can be written as a product of $1-$*degree polynomials and irreducible* $2-$*degree polynomials.*

Theorem

Every polynomial can be written as a product of $1-$*degree polynomials and irreducible* $2-$*degree polynomials.*

Recall that a polynomial $ax^2 + bx + c$ is irreducible if there are no real roots.

Theorem

Every polynomial can be written as a product of $1-$*degree polynomials and irreducible* $2-$*degree polynomials.*

Recall that a polynomial $ax^2 + bx + c$ is irreducible if there are no real roots.
We adopt the following strategy: the polynomial Q in the denominator may be written as a product of the aforementioned polynomials. In that case, the whole fraction is rewritten as a sum of fractions with $1-$ and $2-$ degree polynomials (partial fraction decomposition). This sum may be integrated by methods mentioned in the previous talks.

1st degree polynomials, each with different root

Exercise

- Solve
$$\int \frac{x+1}{x^2+5x+6}\,\mathrm{d}x.$$

1st degree polynomials, some have similar roots

Exercise

- Solve

$$\int \frac{3x^2 - 2x}{(x-1)^2(2x-1)}.$$

2nd degree polynomials

Exercise

- Solve
$$\int \frac{2x+3}{x^2+4x+8}\,\mathrm{d}x.$$
- Solve
$$\int \frac{6x+4}{(x^2+2x+2)(x-1)}\,\mathrm{d}x.$$

Let $\deg P < \deg Q$ and let

$$Q(x) = \alpha_0(x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k}(x^2+p_1x+q_1)^{s_1} \cdot \ldots \cdot (x^2+p_lx+q_l)^{s_l}$$

where the second order polynomials have no real roots and no multiplier divide any other one and all coefficients are integers. Then there are real numbers $A_{11}, \ldots, A_{1r_1}, \ldots, A_{k1}, \ldots, A_{kr_k}$ and $B_{11}, C_{11}, \ldots, B_{1s_1}, C_{1s_1}, \ldots B_{l1}, C_{l1}, \ldots, B_{ls_l}, C_{ls_l}$ such that

$$\begin{aligned} \frac{P(x)}{Q(x)} = \frac{A_{11}}{x-\alpha_1} + \ldots + \frac{A_{1r_1}}{(x-\alpha_1)^{r_1}} + \ldots + \frac{A_{k1}}{(x-\alpha_k)} \\ + \ldots + \frac{A_{kr_k}}{(x-\alpha_k)^{r_k}} + \frac{B_{11}x+C_{11}}{x^2+p_1x+q_1} + \ldots + \frac{B_{1s_1}x+C_{1s_1}}{(x^2+p_1x+q)^{s_1}} \\ + \ldots + \frac{B_{l1}x+C_{l1}}{x^2+p_lx+q} + \ldots + \frac{B_{ls_l}x+C_{ls_l}}{(x^2+p_lx+q_l)^{s_l}} \end{aligned}$$

Exercises

- $$\int \frac{x^2 - 8x + 3}{x^3 - 4x^2 + 3x}\,\mathrm{d}x$$
- $$\int \frac{x^5 + x^3 + x^2 + 2}{x^3 + 1}\,\mathrm{d}x$$
- $$\int \frac{3x^2 + 3x + 3}{x^3 - 3x + 2}\,\mathrm{d}x$$

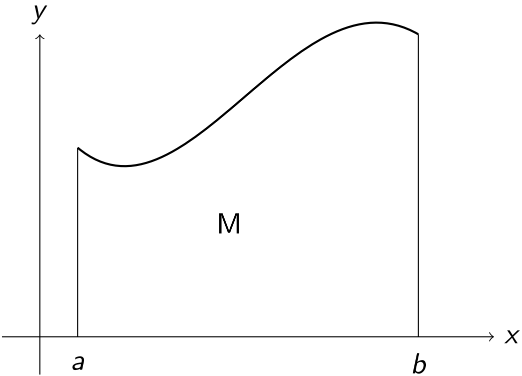
Riemann's and Newton's integral
Riemann's integral

The main aim of this section is to compute the area which is bounded by a graph of function. More precisely, let f be a positive function defined on an interval (a, b). We will try to compute the area of a set

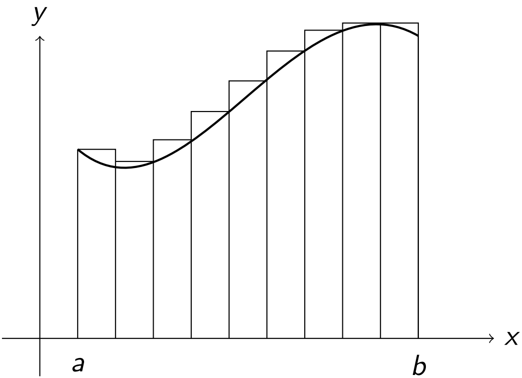
$$M = \{\langle x, y\rangle \in \mathbb{R}^2,\ x \in (a, b),\ 0 < y < f(x)\}. \tag{1}$$

The area is easy assuming $f \equiv c$, $c > 0$. In that case the area is given by $c(b - a)$.
What if f is non-constant?

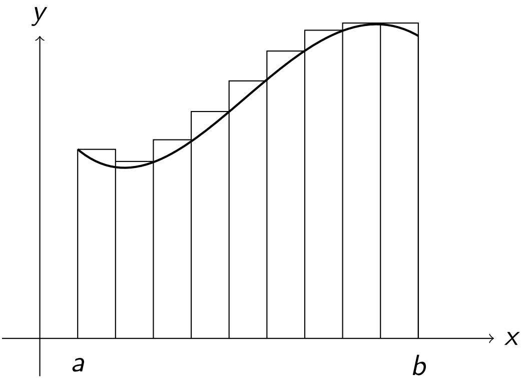
In what follow, we show how to compute an area of the following set:



First, we use rectangles whose union definitely cover the set M:

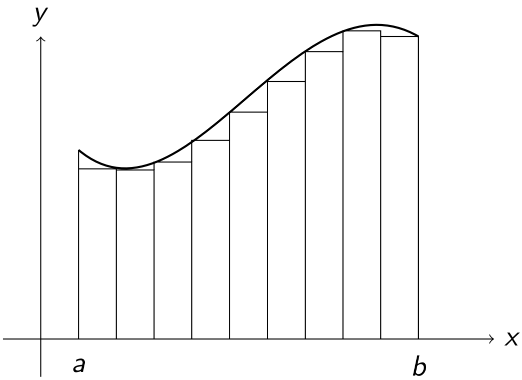


First, we use rectangles whose union definitely cover the set M:



Clearly, the area of M is less than the constructed approximation, however once there will be enough small rectangles, the approximation will be close to the true value.

We can also try to use the following approximation – this time we use maximal rectangle which are inside of the set M



In this case we obtain an area which is less than the area of M.

This idea is summarized in the following definition.

Definition

Let f be a real function defined on $[a, b]$. We define sequences

$$s_n = \sum_{i=1}^{n} \frac{b-a}{n} \min\{f(x), x \in [a + (i-1)(b-a)/n, a + i(b-a)/n]\}$$

$$S_n = \sum_{i=1}^{n} \frac{b-a}{n} \max\{f(x), x \in [a + (i-1)(b-a)/n, a + i(b-a)/n]\}$$

If $\lim s_n = \lim S_n =: s$ then we say that s is the *R*iemann integral of f over (a, b). We write

$$(\mathcal{R}) - \int_a^b f(x) \,\mathrm{d}x = s$$

Exercise

- Compute

$$(\mathcal{R}) - \int_a^b x^2 \,\mathrm{d}x$$

Newton's integral

Definition

Let F be a primitive function of f. Then

$$(\mathcal{N}) - \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

The number $\int_a^b f(x)\,\mathrm{d}x$ is called the Newton integral of f over (a, b).

We use the notation $[F(x)]_a^b$ for the difference $F(b) - F(a)$.

Theorem (The basic theorem of calculus)

Let f be defined on $[a, b]$ and let $(\mathcal{N}) - \int_a^b f(x)\,\mathrm{d}x$ and $(\mathcal{R}) - \int_a^b f(x)\,\mathrm{d}x$ exist. Then

$$(\mathcal{N}) - \int_a^b f(x)\,\mathrm{d}x = (\mathcal{R}) - \int_a^b f(x)\,\mathrm{d}x$$

Let us note several remarks:

- This provides a simple way how to compute an area of the set M defined in the previous lesson.
- As the Riemann and Newton integrals are equal we write simply $\int_a^b f(x)\,\mathrm{d}x$ instead of $(\mathcal{R}) - \int_a^b f(x)\,\mathrm{d}x$ or $(\mathcal{N}) - \int_a^b f(x)\,\mathrm{d}x$.

Exercise

- Compute

$$(\mathcal{N}) - \int_1^2 x^2\,\mathrm{d}x.$$

Definition

Let f defined on (a, b) have the primitive function F. The number

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{x\to b-} F(x) - \lim_{x\to a+} F(x)$$

is called the*generalized* Newton integral of f over (a, b).

Once again, we will use $[F(x)]_a^b$ for $\lim_{x\to a-} F(x) - \lim_{x\to b+} F(x)$.

Exercise

- compute

$$\int_1^4 \sqrt{x}(1+2\sqrt{x})\,\mathrm{d}x$$

- Compute

$$\int_0^2 \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

- Compute

$$\int_3^\infty \frac{1}{x^2}\,\mathrm{d}x$$

- Compute

$$\int_0^\infty e^{-x}\,\mathrm{d}x$$

- Find the

The integral is linear:

$$\int_a^b \alpha f(x) + \beta g(x)\ \mathrm{d}x = \alpha \int_a^b f(x)\ \mathrm{d}x + \beta \int_a^b g(x)\ \mathrm{d}x$$

for integrable f and g and for any constants $\alpha, \beta \in \mathbb{R}$.
The integral is additive:

$$\int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x$$

assuming $a < b < c$ and f integrable.

The integral is monotone: let f and g be two integrable functions such that $f \le g$ for every $x \in (a, b)$. Then

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$$

Exercises

- Compute the area of the region bounded by

$$y = -x^2 + 2x + 8, \; y = 0.$$

- Compute the area of

$$M = \{(x, y) \in \mathbb{R}^2 \, x \geq 0, \, y \geq x^3 \, y \leq 4x\}.$$

- Compute the area of the region inbetween of

$$y = x^2, \; y = 2x^2, \; y = 1.$$

Advanced methods for Newtons integral

Exercise

- Find the area of the region bounded by $y^2 = 2x + 1$ and $y = x - 1$,

Method of substitution, exercises:

- Compute
$$\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}\,\mathrm{d}x.$$
- Compute
$$\int_0^{\pi/2} \sin x \cos^3 x\,\mathrm{d}x.$$
- Compute
$$\int_1^4 \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Method of substitution, exercises:

- Compute
$$\int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}}\,\mathrm{d}x.$$
- Compute
$$\int_0^{\pi/2} \sin x \cos^3 x\,\mathrm{d}x.$$
- Compute
$$\int_1^4 \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

  NONSENSE
- Compute
$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Integration by parts, exercise:

- Compute

$$\int_0^{\pi} x \sin x \,\mathrm{d}x.$$

Recall: The area of the region

$$M = \{(x, y) \in \mathbb{R}^2,\ x \in (a, b),\ g(x) < y < f(x)\}$$

is given as

$$\int_a^b f(x) - g(x)\,\mathrm{d}x.$$

Exercises

- Compute the area of the triangle whose sides are on the lines $y = 3 - x$, $y = 2x$ and $y = \frac{1}{2}x$.
- Find the area the region inbetween of the curve $f(x) = 4 - x^2$ and the x axis where x is from $(-4, 4)$.
- Find the area of the region bounded by two curves $f(x) = x^3$ and $g(x) = x^2 + x$.
- A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

Few words on probability – Discrete probability

Definition

Let X be a real-valued random variable. A *cumulative distribution function* is a function $F(x)$ given as $F(x) = P(X \leq x)$.

- Let there be usual 6 sided fair die. Draw a graph of the cumulative distribution function. What is the probability that the result is lower or equal to 2? What is the probability of an odd result? What is the expectation?

Few words on probability – Discrete probability

Definition

Let X be a real-valued random variable. A *cumulative distribution function* is a function $F(x)$ given as $F(x) = P(X \leq x)$.

- Let there be usual 6 sided fair die. Draw a graph of the cumulative distribution function. What is the probability that the result is lower or equal to 2? What is the probability of an odd result? What is the expectation?
- The odds for the tennis match between Daniil Medvedev and Carlos Alcaraz are 3.23 for Medvedev and 1.38 for Alcaraz. Assuming the probability that Medvedev wins is 30 percent, if we have $100 and we put $40 on Medvedev and $60 on Alcaraz, what is the expected profit? Is there any other distribution of bets which will maximize the profit?

Continuous distribution functions

Example: Two friends want to meet under the tail between 1PM and 2PM. They do not specify the exact time but if one of them come there, he will wait 10 minutes for the second one. What is the probability that they will meet each other?

Continuous distribution functions

Example: Two friends want to meet under the tail between 1PM and 2PM. They do not specify the exact time but if one of them come there, he will wait 10 minutes for the second one. What is the probability that they will meet each other?

Definition

Let X be a real valued random variable. The function $f(x)$ such that $P(X \leq x) = \int_{-\infty}^{x} f(s)\,\mathrm{d}s$ is called a *probability density*.
The *expectation* of a function $g(x)$ is $\int_{-\infty}^{\infty} g(s)f(s)\,\mathrm{d}s =: E(g(X))$.

Continuous distribution functions

Example: Two friends want to meet under the tail between 1PM and 2PM. They do not specify the exact time but if one of them come there, he will wait 10 minutes for the second one. What is the probability that they will meet each other?

Definition

Let X be a real valued random variable. The function $f(x)$ such that $P(X \leq x) = \int_{-\infty}^{x} f(s)\,\mathrm{d}s$ is called a *probability density*.
The *expectation* of a function $g(x)$ is $\int_{-\infty}^{\infty} g(s)f(s)\,\mathrm{d}s =: E(g(X))$.

Exercise

- The tram line number 8 departs from the stop every 10 minutes in the morning. Calculate the probability that you will wait for it for more than 7 minutes in the morning. What is the expected time of waiting?

Exponential distribution

The distribution whose density is given as

$$f(x) = \begin{cases} 0 \text{ for } x < 0 \\ \lambda e^{-\lambda x} \text{ for } x \geq 0. \end{cases}$$

What is its expectation?

Exercise

- The bulb manufacturer Edison knows that the average lifespan of Edison bulbs is 10,000 hours. As part of its promotional campaign, it wants to guarantee a time T during which no more than 3 percents of the bulbs burn out. Determine this time. (Use the exponential distribution to model the lifespan of bulbs.)

Normal distribution

The distribution whose density is given as

$$u(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}.$$

What is the expectation of the random function given by this distribution?

Normal distribution

The distribution whose density is given as

$$u(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}.$$

What is the expectation of the random function given by this distribution? Generalization:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Here μ is the expectation and σ is the standard deviation of the random variable. **Exercises**

- Let U be a real random variable whose density is given by u. Determine the following probabilities:

$$P(U < 1.67),\ P(U > 0.35),\ P(-1.5 < U < 0.5).$$

- Find x such that

$$a)\, P(U < x) = 0.9,\ \ b)\, P(U > x) = 0.8,\ \ c)\, P(-x < U < x) = 0.9.$$

Further exercises

- In the cannery, jars are filled with jam. The weight of the contents of the filled jars follows a normal distribution with a mean of 250 g and a standard deviation of 5 g. The product is within specification if its weight is in the range of 250 ± 8 g. What is the probability that a randomly selected jar is within specification? If four jars are randomly selected, what is the probability that all of them are within specification?
- Men's heights are normally distributed with a mean of 69.0 inches and a standard deviation of 2.8 inches, while women's heights are normally distributed with a mean of 63.6 inches and a standard deviation of 2.5 inches. What percentage of men must duck when walking through a door that is 72 inches high? What percentage of women must duck when walking through a door that is 72 inches high? What door height would allow at least 95% of men to walk through the door without ducking?

Lorenz curve and Gini coefficient

Let f be a density of some random variable. Lorenz curve is a graph of a function

$$L(x) = \frac{\int_{-\infty}^{x} sf(s)\,\mathrm{d}s}{\int_{-\infty}^{\infty} sf(s)\,\mathrm{d}s}.$$

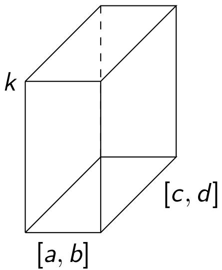
The Gini coefficient is the ration between certain areas.

Example Let us consider an economy with the following population and income statistics:

Population portion	Income portion
0	0
20	10
40	20
60	35
80	60
100	100

Integrals of multivariable functions

Recall, if $f : \mathbb{R} \to \mathbb{R}$ is constant, say $f(x) = c > 0$ for every x, then $\int_a^b f(x)\,\mathrm{d}x = c(b-a)$ is the area of certain rectangle.
Double integral over a rectangle: Assume we have a constant function $f(x,y) \equiv k > 0$ on a set $M = [a,b] \times [c,d]$. What is the volume of a prism $[a,b] \times [c,d] \times [0,k]$? Simple answer is $(b-a)(c-d)k$.



In this particular case we write $\int_{[a,b]\times[c,d]} f(x,y)\ \mathrm{d}x\mathrm{d}y = (b-a)(c-d)k$.

Let M be a rectangle $[a, b] \times [c, d]$ and let $f(x, y)$ be a positive function defined on M. The value of the integral

$$\int_M f(x, y)\ \mathrm{d}x\mathrm{d}y$$

is a volume of a set

$$S = \{(x, y, z) \in \mathbb{R}^3,\ (x, y) \in M,\ 0 \leq z \leq f(x, y)\}.$$

Observation

Let f be a continuous function defined on a rectangle M. Then there is an integral

$$\int_M f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

On measurability

It is not necessary to define integrals only over rectangles. In particular, the set M can be 'almost arbitrary' and then the meaning of integral is the same as in the previous slide. The only condition is that the integral $\int_M 1 \, \mathrm{d}x\mathrm{d}y$ has value (and it might be even infinity). Such sets are called *measurable sets* and we will not define them in the scope of this class. Let me just mention that not every set is measurable. On the other hand, it is very difficult to construct a non-measurable set. All sets appearing in this class are measurable.

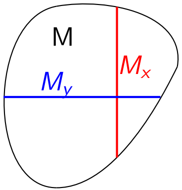
Similarly, one can define integral for a larger class of function than just continuous functions. This class is called *measurable functions*. Similarly as before, it is very difficult to construct a measurable function and all functions appearing in this class are measurable.

Definition

Let $M \subset \mathbb{R}^2$. We define a *vertical cross-section* as

$$M_x = \{y \in \mathbb{R},\ (x, y) \in M\}.$$

Similarly, a *horizontal cross-section* as $M_y = \{x \in \mathbb{R},\ (x, y) \in M\}$.



Exercises

Write the horizontal and vertical cross-sections of

- the triangle with vertices at $(2, 0)$, $(0, 2)$, and $(-2, -2)$.
- the set $M = \{(x, y) \in \mathbb{R}^2,\ y \geq x^2,\ y \leq (x-2)^2,\ y \leq (x+2)^2\}$.

Theorem

(Fubini) Let $M \subset \mathbb{R}^2$ is a measurable set and $f : M \to \mathbb{R}$ be a measurable function. Then

$$\int_M f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{M_x} f(x,y)\,\mathrm{d}y \right)\,\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{M_y} f(x,y)\,\mathrm{d}x \right)\,\mathrm{d}y.$$

assuming that the integral on the left hand side is well defined.

Exercise

- Compute

$$\int_M 5x^2y - 2y^3\,\mathrm{d}x\mathrm{d}y,\ M = [2,5] \times [1,3].$$

Further properties:

Exercise

- Compute

$$\int_M 2xe^y \,\mathrm{d}x\mathrm{d}y, \;\; M = [0,2] \times [0,1].$$

Further properties:

Exercise

- Compute

$$\int_M 2xe^y \,\mathrm{d}x\mathrm{d}y, \;\; M = [0,2] \times [0,1].$$

Observation

Let $f(x,y) = g(x)h(y)$ *and let* $M = [a,b] \times [c,d]$. *Then*

$$\int_M f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_a^b g(x)\,\mathrm{d}x\mathrm{d}y \int_c^d h(y)\,\mathrm{d}y.$$

Observation

The muldi-dimensional integral possess the following properties:

- *Linearity: let f and g be measurable function defined on a measurable set $M \subset \mathbb{R}^2$ and let $\alpha \in \mathbb{R}$. Then*

$$\int_M \alpha f(x,y) + g(x,y)\,\mathrm{d}x\mathrm{d}y = \alpha \int_M f(x,y)\,\mathrm{d}x + \int_M g(x,y)\,\mathrm{d}y.$$

- *If the f is nonnegative, then*

$$\int_M f(x,y)\,\mathrm{d}x\mathrm{d}y \geq 0.$$

- *Let M itself be a union of n measurable subsets, i.e. $M = M_1 \cup M_2 \cup \ldots \cup M_n$. Then*

$$\int_M f(x,y)\,\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{n} \int_{M_i} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Further exercises

- Change the order of integration in

$$\int_{-1}^{1} \left(\int_{-1+y^2}^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

- Compute

$$\int_M \frac{x}{y+1}\,\mathrm{d}x\mathrm{d}y,\ M \text{ is a triangle with vertices } (0,-1),\ (-1,2),\ (1,2).$$

- Compute

$$\int_M (x+y)^2 + y\,\mathrm{d}x\mathrm{d}y$$

where M is the square with vertices $A = (2,0)$, $B = (0,2)$, $C = (-2,0)$, and $D = (0,-2)$.

Further exercises

- Change the order of integration in

$$\int_{-1}^{1}\left(\int_{-1+y^2}^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x\right)\,\mathrm{d}y.$$

- Compute

$$\int_M \frac{x}{y+1}\,\mathrm{d}x\mathrm{d}y,\ \ M \text{ is a triangle with vertices } (0,-1),\ (-1,2),\ (1,2).$$

- Compute

$$\int_M (x+y)^2 + y\,\mathrm{d}x\mathrm{d}y$$

  where M is the square with vertices $A=(2,0)$, $B=(0,2)$, $C=(-2,0)$, and $D=(0,-2)$.

Is there a better way of computing integrals such as the last one? Yes, there is.

Change of variables

The last exercise we did:

$$\int_M (x+y)^2 + y \,\mathrm{d}x\mathrm{d}y, M \text{ is a square with vertices } A=(0,-2),$$

$$B=(2,0),\ C=(0,2),\ D=(-2,0).$$

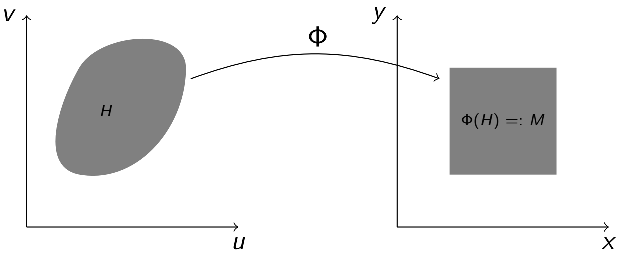
Change of variables

The last exercise we did:

$$\int_M (x+y)^2 + y \,\mathrm{d}x\mathrm{d}y, M \text{ is a square with vertices } A = (0,-2),$$
$$B = (2,0),\ C = (0,2),\ D = (-2,0).$$

Recall, that the one-dimensional substitution method works in the following way

$$\int_a^b f(t)\,\mathrm{d}t = \int_\alpha^\beta f(\varphi(x))\varphi'(x)\,\mathrm{d}x.$$



Mapping $\Phi(u,v) = (\varphi(u,v), \psi(u,v))$, $x = \varphi(u,v)$, $y = \psi(u,v)$.

Definition A mapping $\Phi : H \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

- $\Phi \in C^1$,
- Φ is an injection,
- The Jacobian matrix of Φ is regular,

is called a *regular mapping*.

Definition Let $\Phi : H \subset \mathbb{R}^2 \mapsto \mathbb{R}^2$ have components $\varphi(u,v)$ and $\psi(u,v)$. Then the *Jacobian matrix* of Φ is a matrix

$$J\Phi(u,v) = \begin{pmatrix} \frac{\partial\varphi}{\partial u}(u,v) & \frac{\partial\varphi}{\partial v}(u,v) \\ \frac{\partial\psi}{\partial u}(u,v) & \frac{\partial\psi}{\partial v}(u,v) \end{pmatrix}.$$

Its determinant is then called the *Jacobian determinant*.

Definition Let $\Phi : H \subset \mathbb{R}^2 \mapsto \mathbb{R}^2$ have components $\varphi(u, v)$ and $\psi(u, v)$. Then the *Jacobian matrix* of Φ is a matrix

$$J\Phi(u, v) = \begin{pmatrix} \frac{\partial\varphi}{\partial u}(u, v) & \frac{\partial\varphi}{\partial v}(u, v) \\ \frac{\partial\psi}{\partial u}(u, v) & \frac{\partial\psi}{\partial v}(u, v) \end{pmatrix}.$$

Its determinant is then called the *Jacobian determinant*.

Theorem Let $f(x, y)$ is a measurable function on $D \subset \mathbb{R}^2$ and let $\Phi = (\varphi, \psi) : H \subset \mathbb{R}^2 \mapsto M$ is a regular mapping. Then

$$\int_M f(x, y)\, \mathrm{d}x\mathrm{d}y = \int_H f(\varphi(u, v), \psi(u, v))|\det J\Phi(u, v)|\, \mathrm{d}u\mathrm{d}v$$

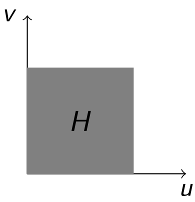
The role of the Jacobian determinant: Consider a mapping

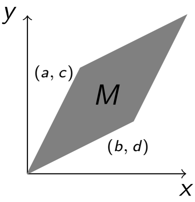
$$x = au + bv =: \varphi(u, v)$$
$$y = cu + dv =: \psi(u, v).$$

Here we have that

$$J\Phi(u, v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let $H = (0, 1) \times (0, 1)$. Then M is a parallelogram with sides represented by vectors (a, c) and (b, d).





The area of H is $\int_H 1 \,\mathrm{d}u\mathrm{d}v = 1$ and the area of M is $\int_M 1 \,\mathrm{d}x\mathrm{d}y = \left|\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|$. Indeed, the area of the parallelogram is equal to $S = \sin\alpha \|(a,c)\|\|(b,d)\|$ and we may compute

$$\begin{aligned} S^2 = \sin^2\alpha \|(a,c)\|^2\|(b,d)\|^2 &= \left(1-\cos^2\alpha\right)\|(a,c)\|^2\|(b,d)\|^2 \\ &= \|(a,c)\|^2\|(b,d)\|^2 - ((a,c)\cdot(b,d))^2 \\ &= (a^2+c^2)(b^2+d^2) - (ab+cd)^2 = a^2d^2 + c^2b^2 - 2abcd \\ &= (ad-bc)^2. \end{aligned}$$

Therefore, there has to be a factor $|\det J\Phi|$ in order to get

$$\int_M 1 \,\mathrm{d}x\mathrm{d}y = \int_H 1 |\det J\Phi| \,\mathrm{d}u\mathrm{d}v.$$

Exercises

- Let compute the integral from the beginning, i.e.,

$$\int_M (x+y)^2 + y \,\mathrm{d}x\mathrm{d}y$$

where $M = \{(x,y) \in \mathbb{R}^2,\ -2 \le x+y \le 2,\ -2 \le x-y \le 2\}$.

- Use the transformation $\Phi(u,v) = (u, u+v)$ to compute

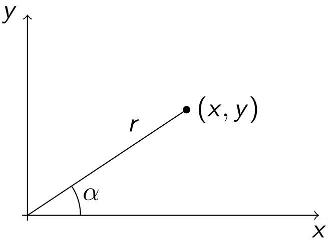
$$\int_1^2 \int_{x+2}^{x+3} \frac{1}{\sqrt{xy - x^2}} \,\mathrm{d}x\mathrm{d}y$$

- Evaluate

$$\int_M \frac{y^2}{x} \,\mathrm{d}x\mathrm{d}y$$

where M is the region between parabolas $x = 1 - y^2$ and $x = 3(1 - y^2)$. Use the transformation $x = v(1 - u^2)$ and $y = u$.

Polar coordinates



We have

$$x = r\cos\alpha$$
$$y = r\sin\alpha.$$

Exercise

- Compute the Jacobian determinant of the above mapping.

Exercises

- Compute the volume of the ball centered at origin whose radius is R^2.
- Compute an area of the set M which is given by the following conditions:

$$(x^2 + y^2)^2 \leq 2xy$$

- Compute

$$\int_{-\infty}^{\infty} e^{-ax^2} \, \mathrm{d}x$$

where a is a positive constant.

Adjusted polar coordinates

Let compute an area of an ellipse which is given as

$$M = \left\{ (x, y) \in \mathbb{R}^2,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

for some positive reals a and b.

Several concluding examples

- The total cost of a company is given as

$$TC(Q) = 4Q^2 + 3Q + 1.$$

  Find the mean value of $TC(Q)$ on an interval $[3, 6]$ and determine the point where it is achieved.

- A company invested \$ 750 000 in news, more efficient machines that will produce product in its factory. The savings generated by the deployment of the new machines are given as

$$S(t) = 400\,000e^{-0.4t}$$

  where t is time in years. How long will it take for the investment to return to the company?

- Find $a \in \mathbb{R}$ such that
$$f(x) = \chi_{[0,a]} e^{\frac{x}{2}}$$
is a probability density.
- Rewrite the integral
$$\int_2^3 \sqrt{\frac{x+1}{x-1}}\,\mathrm{d}x$$
as a double integral and compute it.