UCT Mathematics

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These lecture notes contain mathematical knowledge needed to pass through math exam at the University of Chemistry and Technology. They are released online and they are available for free.
On the other hand, my work on this text is still not finished and the lecture notes will be updated several times during the semester. Thus, this text may contain mistakes. In case you find any, let me know.

Contents

1 Numbers, sets, and functions

1.1 Logic

A *proposition* is such sentence that we can decide about its correctness, i.e., whether it is true or false. For example:
– 'three plus four' is not a proposition,
– 'three plus four is six' is a proposition (obviously wrong). This proposition is atomic (elementary) – it cannot be decomposed. – 'three plus four is seven and one plus one is three' is a proposition as well, however, this proposition is not atomic since it can be decomposed into a proposition 'three plus four is seven', into another proposition 'one plus one is three' and a connective 'and'.

How to make non-atomic propositions
Propositions may be joined into new proposition by using logical connectives:

- conjunction - and - &: 'three plus four is seven and one plus one is three' is an example of a conjunction of two propositions, proposition $A =$'three plus four is seven' and proposition $B =$'one plus one is three'. It may be written as $A\&B$. The whole conjunction is false. Nevertheless, if we replace B by $C =$'one plus one is two', then $A\&C$ will be true – the conjunction is true only if both propositions are true.

- disjunction - or - $\vee$: Using the same notation as above, we understand $A \vee B$ as 'three plus four is seven or one plus one is three'. This time, the proposition $A \vee B$ is true – the disjunction is true once there is at least one true proposition.

- implication - if ... then - $\Rightarrow$: 'if sun shines then it is hot' – here we have two elementary propositions $D =$'sun shines' and $E =$'it is hot'. The implication $D \Rightarrow E$ is false only in case the sun shines and, simultaneously, it is not hot. The implication is true in all other cases.

- equivalence - if and only if - $\Leftrightarrow$

- negation - it is not true that ... - $\neg$: 'it is not true that sun shines', or, with the above notation, $\neg D$. Note that this particular negation might be abbreviated as 'the sun does not shine'. It holds that $\neg\neg A = A$.

The summary is provided by the following table

A	B	$A\&B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$	$\neg A$
true	true	true	true	true	true	false
true	false	false	true	false	false	false
false	true	false	true	true	false	true
false	false	false	false	true	true	true

- Some rules (including the De Morgan laws):
$A \Rightarrow B$ is the same as $(\neg A) \vee B$

$\neg(A \vee B)$ is the same as $(\neg A)\&(\neg B)$
$\neg(A\&B)$ is the same as $(\neq A) \vee (\neg B)$
as a result of the previous lines we deduce that $\neg(A \Rightarrow B)$ is $A\&\neg B$.

Quantifiers

Existential quantifier $\exists$ is read as 'there is' or 'there exists'. For example, 'there exists a natural number n such that $2n = 5$' can be written by use of symbols as $\exists n \in \mathbb{N}, 2n = 5$ (here $\mathbb{N}$ denotes a set of all natural numbers – see the next subsection). We just remark that this statement is false.

Universal quantifier $\forall$ is read as 'for all' or 'every'. For example 'every unicorn can breath under water'. The first two words of this sentence might be shortened to $\forall u \in U$ where U denotes a set of all unicorns. Let us remark that the above statement is true – every proposition is true assuming it tackles all individuals from an empty set, here we tacitly assume that unicorns do not exist.

1.2 Sets

The sets are given by one of the following ways:

- list of elements: $M = \{1, 2, 3, 4\}$ is a set containing numbers 1, 2, 3 and 4.
- a condition (or more conditions): $M = \{w \text{ is a word containing exactly five letters}\}$ or $M = \{w \text{ is a word containing exactly five letters}, w \text{ is a noun}\}$.

Definition 1.1. *Let X and Y be two sets. By $X \cup Y$ we denote a* union *of sets X and Y which is a set containing elements of both sets, i.e.,*

$$X \cup Y = \{x,\ (x \in X) \vee (x \in Y)\}.$$

By $X \cap Y$ we denote an intersection *of sets X and Y which is a set consisting of elements belonging simultaneously to both sets, i.e.,*

$$X \cap Y = \{x,\ (x \in X)\&(x \in Y)\}.$$

The Cartesian product *$X \times Y$ is a set of all ordered couples such that the first component belongs to X and the second to Y. Namely,*

$$X \times Y = \{\langle x, y\rangle,\ (x \in X)\&(y \in Y)\}.$$

We say that X is a subset *of Y if every element of X is in Y. The notation is $X \subset Y$ and we may write*

$$X \subset Y \Leftrightarrow ((x \in X) \Rightarrow (x \in Y)).$$

Sets X and Y are equal *if $X \subset Y$ and simultaneously $Y \subset X$.*
Let $X \subset Y$. By $Y \setminus X$ we understand a set of all elements in Y which are not in X, i.e.,

$$Y \setminus X = \{(y \in Y)\&(y \notin X)\}.$$

Hereinafter, the empty set is denoted by $\emptyset$.

1.3 Numbers

We will use the following notation for numbers: $\mathbb{N}$ stands for natural numbers, $\mathbb{Z}$ denotes integers, $\mathbb{Q}$ is a set of all rational numbers and $\mathbb{R}$ denotes the set of all real numbers. Namely:

$$\mathbb{N} = \{1, 2, \ldots\}$$

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$\mathbb{Q} = \left\{\ldots, \frac{1}{2}, \frac{-3}{2}, \frac{5}{2}, \ldots\right\}$$

$\mathbb{Q}$ is a field. Namely, there are two operations $+$ and $\cdot$ fulfilling

- $\forall x, y, z \in \mathbb{Q}$, $x + (y + z) = (x + y) + z$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity)
- $\forall x, y \in \mathbb{Q}$, $x + y = y + x$, $x \cdot y = y \cdot x$ (commutativity)
- $\exists 0 \in \mathbb{Q}$, $\forall x \in \mathbb{Q}$, $x + 0 = x$ (there is null)
- $\exists 1 \in \mathbb{Q}$, $\forall x \in \mathbb{Q}$, $x \cdot 1 = x$ (there is one)
- $\forall x \in \mathbb{Q}$, $\exists - x \in \mathbb{Q}$, $x + (-x) = 0$ (there is an opposite number)
- $\forall x \in \mathbb{Q} \setminus \{0\}$, $\exists x^{-1}$, $x \cdot x^{-1} = 1$ (there is an inverse number)
- $\forall x, y, z \in \mathbb{Q}$, $x \cdot (y + z) = x \cdot y + x \cdot z$ (distributivity)

The set $\mathbb{R}$ is also a field, which is totally ordered and it containes supremum and infimum of every of its subset. Therefore, to properly states basic properties of all real numbers $\mathbb{R}$ we first define a totally ordered set as well as supremum and infimum:

Definition 1.2. *We say that a set X is* totally ordered *if there is a relation $\leq$ fulfilling*

- $\forall x, y \in X,\ (x \leq y) \vee (y \leq x)$.
- $\forall x, y \in X,\ ((x \leq y) \& (y \leq x)) \Rightarrow x = y$.
- $\forall x, y, z \in X,\ ((x \leq y) \& (y \leq z)) \Rightarrow (x \leq z)$.

Further, we define relation $<$ as $x < y \Leftrightarrow (x \leq y \& x \neq y)$.

$\mathbb{R}$ is a totally ordered field which, moreover, satisfies

- $\forall x, y, z \in \mathbb{R}$, $(x < y) \Rightarrow (x + z < y + z)$
- $\forall x, y \in \mathbb{R}$ and $z > 0$, $(x < y) \Rightarrow (z \cdot x < z \cdot y)$

Remark 1.1. *Simply, $x \geq y$ is the same as $y \leq x$ and $x > y$ is the same as $y < x$.*

Definition 1.3. *Let $A \subset \mathbb{R}$. We define a* supremum *(or least upper bound, abbreviated as LUB) of A,* $\sup A$, *as a number $M \in \mathbb{R}$ fulfilling*

$$\forall x \in A,\ (x \leq M) \& (\forall \varepsilon > 0, \exists x \in A,\ x + \varepsilon > M)$$

Similarly, we define infimum *(or greatest lower bound, abbreviated as GLB) of A,* $\inf A$, *as a number $m \in \mathbb{R}$ fulfilling*

$$\forall x \in A,\ (x \geq m) \& (\forall \varepsilon > 0, \exists x \in A,\ x - \varepsilon < m).$$

Define $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. Previous definition allows to state the last property of real numbers which is: $\forall A \subset \mathbb{R}$, $\exists M \in \mathbb{R}^*$, $M = \sup A$. That is the way how we get the *extended real numbers* – denoted by $\mathbb{R}^*$ – since we add also numbers $+\infty$ and $-\infty$ – however that was not intended. In order to get the demanded field of numbers we remove $+\infty$ and $-\infty$ as the very last step. Thus $\mathbb{R} = \mathbb{R}^* \setminus \{+\infty, -\infty\}$.

It is worth to mention that rational numbers do not posses this last property. Namely, $\sqrt{2}$ is real number, since it can be defined as

$$\sqrt{2} = \sup\{x, x^2 \le 2\}.$$

On the other hand, $\sqrt{2}$ is not a rational number. Indeed, let $\sqrt{2} = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that p and q do not have a common divisor (and thus it cannot be simplified). Then $(\frac{p}{q})^2 = 2$ which implies $p^2 = 2q^2$ and 2 is a divisor of p which can be written as $p = 2l$ for some $l \in \mathbb{Z}$. We put it into the last equality to get $4l^2 = 2q^2$ yielding $2l^2 = q^2$ and 2 is a divisor of q. Thus p and q have a common divisor 2 which is a contradiction with our assumption.

Definition 1.4. *Let* a, $b \in \mathbb{R}^*$, $a < b$. *An* open interval (a, b) *is defined as* $(a, b) = \{x \in \mathbb{R}, a < x < b\}$. *Let* a, $b \in \mathbb{R}$, $a < b$. *A* closed interval $[a, b]$ *is defined as* $[a, b] = \{x \in \mathbb{R}, a \le x \le b\}$. *Further, we define* half-open interval *as follows: Let* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^*$ *be such that* $a < b$. *Then* $[a, b) = \{x \in \mathbb{R}, a \le x < b\}$. *Let* $a \in \mathbb{R}^*$ *and* $b \in \mathbb{R}$. *Then* $(a, b] = \{x \in \mathbb{R}, a < x \le b\}$.

1.4 Few words about proofs

A mathematical theorem (lemma, observation) are usually of the form $A \Rightarrow B$ where A denotes the assumptions of a theorem and B denotes the claims of the theorem. The methods of proof of such implication is the following:

- Direct. To prove $A \Rightarrow B$, we present a set of implications which starts from A and end up in B.
  Example: Let $a > 1$, then $a^2 > 1$. Proof: $(a > 1) \Rightarrow (a > 0) \Rightarrow (a^2 > a > 1) \Rightarrow (a^2 > 1)$.
- Indirect. Rather than proving $A \Rightarrow B$, we prove $\neg B \Rightarrow \neg A$.
  Example: Let $a, b \in \mathbb{R}$ and let $ab = 0$. Then either $a = 0$ or $b = 0$. Proof: we show that $(a \neq 0)\&(b \neq 0)$ implies $ab \neq 0$. Let $a > 0$ and $b > 0$. Then $ab > 0$. In other cases we proceed similarly, for example if $a < 0$ and $b > 0$, then we use the previous argument for $-a$ and b.
- Contradiction. Instead of proving $A \Rightarrow B$, we show that $A \& \neg B$ yields contradiction and thus cannot occur. For example a claim $\sqrt{2} \notin \mathbb{Q}$ which was presented in the previous subsection.
- Mathematical Induction – special kind of proof, the rest of this subsection is devoted to this.

Mathematical induction
is a method how to prove an assertion $V(n)$ for every $n \in \mathbb{N}$. (For example, let $n \in \mathbb{N}$ and let $V(n)$ be 'it holds that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$'.)
Math induction helps to prove that $V(n)$ holds for every $n \in \mathbb{N}$. It consists of two steps:

1. $V(1)$ holds.
2. for every $k \in \mathbb{N}$ it holds that $V(k) \Rightarrow V(k+1)$.

Example:
We prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. First, we show the validity of this equality for $n = 1$. In this case we have

$$L = 1 = \frac{1 \cdot 2}{2} = R.$$

To verify the second step, assume that for an arbitrary $k \in \mathbb{N}$ the assertion holds true. We intent to prove that

$$\left(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\right) \Rightarrow \left(\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}\right)$$

To prove the last equality, let start with its left hand side and show it is equal to the right hand side (by using the assumption). We have

$$L = \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2} = R.$$

1.5 Mappings

Definition 1.5. *Let $f \subset (X \times Y)$ be a subset which fulfills for every $x \in X$ and $y_1,\ y_2 \in Y$ that*

$$((\langle x, y_1\rangle \in f)\&(\langle x, y_2\rangle \in f)) \Rightarrow (y_1 = y_2).$$

Then we say that f is a mapping *which maps X to Y. We write $f : X \to Y$. A usual notation for $\langle x, y\rangle \in f$ is $f(x) = y$ or $f : x \mapsto y$.*

A domain *is a set of all $x \in X$ for which there exists y such that $f(x) = y$. The domain of f is denoted by* Dom f. *The set of all $y \in Y$ for which there exists $x \in X$ such that $f(x) = y$ is called* range. *It is denoted by* Ran f.

Let $A \subset$ Dom f. An image of A (denoted by $f(A)$) is a set in Ran f *defined as*

$$f(A) = \{y \in Y,\ \exists x \in A,\ y = f(x)\}.$$

Let $B \subset$ Ran f. A preimage of B (denoted by $f^{-1}(B)$) is a set in Dom f *defined as*

$$f^{-1}(B) = \{x \in X,\ \exists y \in B,\ y = f(x)\}.$$

Remark 1.2. *Usually, if X and Y are number sets ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$), then f is called a* function. *Nevertheless, we will often use the term 'function' also for mappings.*

Example: Let $f = \{\langle 3, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$. We have Dom $f = \{1, 2, 3\}$, Ran $f = \{1, 2\}$. On the other hand, let $g = \{\langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$. Now g is not a function because we have one value of x ($x = 2$) which is mapped to two different values of y (either $y = 1$ or $y = 2$). This contradicts the very first property of the definition.

Observation 1.1. *For every $A,\ B \subset$ Dom f it holds that*

$$f(A \cup B) = f(A) \cup f(B)$$

Proof. It holds that

$$\begin{aligned}(y \in f(A \cup B)) \Rightarrow (\exists x \in (A \cup B),\ y = f(x)) \Rightarrow ((\exists x \in A,\ y = f(x)) \vee (\exists x \in B,\ y = f(x)))\\ \Rightarrow ((y \in f(A)) \vee (y \in f(B))) \Rightarrow (y \in f(A) \cup f(B))\end{aligned}$$

and we have just proven that $f(A \cup B) \subset (f(A) \cup f(B))$.
On the other hand

$$(y \in f(A) \cup f(B)) \Rightarrow ((y \in f(A)) \vee (y \in f(B))) \Rightarrow ((\exists x \in A,\ y = f(x)) \vee (\exists x \in B,\ y = f(x)))$$
$$\Rightarrow (\exists x \in (A \cup B),\ y = f(x)) \Rightarrow (y \in f(A \cup B))$$

which yields $(f(A) \cup f(B)) \subset f(A \cup B)$. This concludes the proof. □

Definition 1.6. *A function $f : X \mapsto Y$ is said to be*

- injective *if* $\forall x_1,\ x_2 \in \text{Dom } f,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$*,*
- surjective *if* $\text{Ran } f = Y$*,*
- bijective *if it is* surjective *and* injective*.*

We use a term injection *(resp.* surjection *or* bijection*) for injective (resp. surjective of bijective) function.*

Example: Let consider the mapping from the previous example, i.e., $f = \{\langle 3, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$. This function is not injective since $f(1) = 2$ as well as $f(2) = 2$. On the other hand, when taking $Y = \{1, 2\}$, then f is sufjective.

Definition 1.7. *Let $f : X \to Y$ and let $g : Y \to Z$ be such that* $\text{Ran } f \subset \text{Dom } g$*. Then a* composition *of functions g and f is a function $g \circ f : X \to Z$ defined as*

$$(g \circ f)(x) = g(f(x)).$$

If there is a function $g : Y \to X$ such that $\text{Dom } f = \text{Ran } g$, $\text{Dom } g = \text{Ran } f$*, $(g \circ f)(x) = x$ for all $x \in$* $\text{Dom } f$ *then g is called an* inverse function *to f and we denote it by f^{-1}. An* invertible *function is a function for which there exists the inverse function.*

Example: Take the function f from the previous example and consider a function h given as

$$h = \{\langle 1, 5\rangle, \langle 2, 8, \rangle\}.$$

Since $\text{Dom } h = \{1, 2\} = \text{Ran } f$, we may write down a function $h \circ f$ (or, equivalently $h(f(x))$. We have

$$h(f(3)) = 5,\ h(f(1)) = 8,\ h(f(2)) = 8.$$

The function f from the previous example is not invertible since it is not injective. Take a function j defined as

$$j = \{\langle 1, 4\rangle, \langle 2, 1\rangle, \langle 3, 7\rangle, \langle 4, 10\rangle\}.$$

The function h is injective and it is surjective assuming $Y = \{1, 4, 7, 10\}$. Thus there exists j^{-1} and it is a function

$$j^{-1} = \{\langle 1, 2\rangle, \langle 4, 1\rangle, \langle 7, 3\rangle, \langle 10, 4\rangle\}.$$

Observation 1.2. *It holds that* $\text{Dom } f = \text{Ran } f^{-1}$ *and* $\text{Ran } f = \text{Dom } f^{-1}$ *whenever f is an invertible function.*

Proof. Obvious. □

Recall that a function $f(x) = x$ is often called *identity* and that not every function has its inverse. Moreover, $f \circ g$ is also an identity.

Observation 1.3. *Let $f : X \to Y$, Dom $f = X$. The inverse function f^{-1} exists if and only if f is injective.*

Proof. Let f be injective. Thus, for every $y \in \text{Ran } f$ there exists only one $x \in \text{Dom } f$ such that $y = f(x)$. It suffices to define $f^{-1}(y) = x$.

Let f be not injective. There exist $x_1,\ x_2 \in \text{Dom } f$ and $y \in \text{Ran } f$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$. Let $f^{-1}(y) = x_1$ – this is necessary to have $f^{-1}(f(x_1)) = x_1$. Then $f^{-1}(f(x_2)) = f^{-1}(y) = x_1 \neq x_2$ and thus f^{-1} is not an inverse function. □

Definition 1.8. *An* indicator function *of a set $A \subset X$ is a function $f : X \mapsto \{0, 1\}$,* Dom $f = X$ *fulfilling $f(x) = 1$ if and only if $x \in A$. Such function is denoted by χ_A.*

Definition 1.9. *We say that $f : X \mapsto \mathbb{R}$ is* bounded from above *if there is $M \in \mathbb{R}$ such that $f(x) \leq M$ for each $x \in$* Dom f*. It is* bounded from below *if there is $m \in \mathbb{R}$ such that $f(x) \geq m$ for every $x \in$* Dom f*. We say that f is* bounded *if f is bounded from above and from below.*

1.6 Exercises

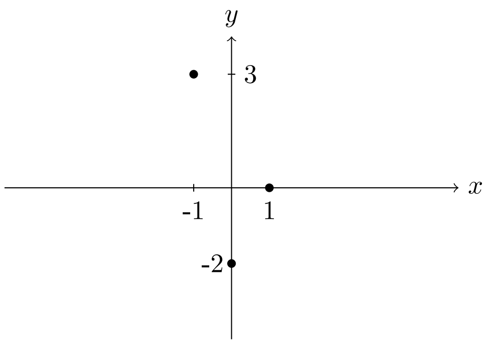
1. Show that $1 > 0$.
2. Show that $\sup(0, 2) = \sup[0, 2] = 2$.
3. Find $\sup A$ and $\inf A$ of $A = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ (i.e., a set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$).
4. Which of these subsets of $\mathbb{N} \times \mathbb{N}$ is a function?
   (a) $f = \{\langle 1, 5\rangle, \langle 2, 4\rangle, \langle 1, 3\rangle\}$
   (b) $g = \{\langle 1, 2\rangle, \langle 5, 3\rangle, \langle 10, 1\rangle\}$
   (c) $h = \{\langle 3, 3\rangle, \langle 4, 3\rangle, \langle 7, 7\rangle, \langle 10, 3\rangle\}$
5. Consider a function h defined in the previous exercise. Write Dom h and Ran h.
6. Does the following modification of Observation 1.1
$$\forall A, B \subset \text{Dom } f,\ f(A \cap B) = f(A) \cap f(B)$$
   hold? If yes, prove it. If no, try to think for which functions does it hold.
7. Let $f,\ g : \mathbb{N} \mapsto \mathbb{N}$ be defined as
$$\begin{aligned} f &= \{\langle 2, 2\rangle,\ \langle 3, 2\rangle,\ \langle 4, 6\rangle,\ \langle 1, 3\rangle\} \\ g &= \{\langle 2, 3\rangle,\ \langle 3, 2\rangle,\ \langle 6, 2\rangle\} \end{aligned}$$
   Write $f \circ g$ and $g \circ f$.
8. Let f be an invertible function. Show that f^{-1} is determined uniquely.

2 Real functions

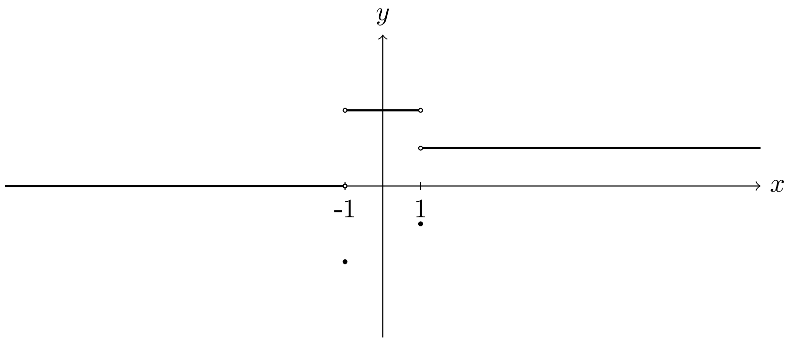
By a *real function* we mean a function $f : \mathbb{R} \mapsto \mathbb{R}$.

Definition 2.1. *A* graph *of a function f is a subset of plane consisting of points $\langle x, f(x) \rangle$ where $x \in \operatorname{Dom} f$.*

Consider a function $f = \{\langle 1, 0\rangle,\ \langle -1, 3\rangle,\ \langle 0, -2\rangle\}$. Its graph looks as follows



It is worth pointing out that $\operatorname{Dom} f = \{-1, 0, 1\}$ and $\operatorname{Ran} f = \{-2, 0, 3\}$.
A graph of function $f = 2\chi_{(-1,1)} - 2\chi_{\{-1,1\}} + \chi_{[1,\infty)}$ is



Definition 2.2. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ and $I \subset \operatorname{Dom} f$ be an interval. We say that f is on I*

- increasing *if* $\forall x_1,\ x_2 \in I,\ (x_1 < x_2) \Rightarrow f(x_1) < f(x_2)$,
- decreasing *if* $\forall x_1,\ x_2 \in I,\ (x_1 < x_2) \Rightarrow f(x_1) > f(x_2)$,
- non-decreasing *if* $\forall x_1,\ x_2 \in I,\ (x_1 < x_2) \Rightarrow f(x_1) \leq f(x_2)$,
- non-increasing *if* $\forall x_1,\ x_2 \in I,\ (x_1 < x_2) \Rightarrow f(x_1) \geq f(x_2)$.

If f posses one of these properties we will say that f is monotone.

Definition 2.3. *A function $f : \mathbb{R} \mapsto \mathbb{R}$ is called* periodic*, if there is a number $l > 0$ such that $f(x) = f(x + l)$ for all $x \in \mathbb{R}$. The least number l with that property is called a* period *of a function f and f is then $l-$periodic.*

Definition 2.4. *A function* $f : \mathbb{R} \mapsto \mathbb{R}$ *is said to be* continuous *at a point* $x_0 \in \mathrm{Dom}\ f$ *if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (x_0 - \delta, x_0 + \delta) \cap \mathrm{Dom}\ f,\ |f(x) - f(x_0)| < \varepsilon.$$

A function $f : \mathbb{R} \mapsto \mathbb{R}$ *is said to be* left-continuous *(resp.* right-continuous*) at a point* $x_0 \in \mathrm{Dom}\ f$ *if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (x_0 - \delta, x_0) \cap \mathrm{Dom}\ f,\ |f(x) - f(x_0)| < \varepsilon$$
$$(\textit{resp. } \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (x_0, x_0 + \delta) \cap \mathrm{Dom}\ f,\ |f(x) - f(x_0)| < \varepsilon)$$

We say that f *is continuous on a set* $S \subset \mathbb{R}$ *if it is continuous at all of its points.*

We define a function $\mathrm{sgn}\ (x) = \chi_{[0,\infty)} - \chi_{(-\infty,0)}$ – the function is called 'signum' or 'sign', it is equal to -1 for x negative and 1 otherwise. This function is not continuous at $x_0 = 0$. However, it is right-continuous at 0. Indeed, let $\varepsilon = \frac{1}{2}$. Then for every $\delta > 0$, $-\frac{\delta}{2} \in (-\delta, \delta)$ and

$$\left|\mathrm{sgn}\ \left(-\frac{\delta}{2}\right) - \mathrm{sgn}\ (0)\right| = |-1-1| = 2 > \frac{1}{2}.$$

On the other hand, for every $\varepsilon > 0$ we can state (for example) $\delta = \varepsilon$ and then for every $x \in (0, \delta)$ it holds that $\mathrm{sgn}\ (x) = 1 = \mathrm{sgn}\ (0)$ and thus $|\mathrm{sgn}\ (x) - \mathrm{sgn}\ (0)| = 0 < \varepsilon$.

Before we go on let us recall the triangle inequality

$$|a + b| \le |a| + |b|$$

which holds true for all $a, b \in \mathbb{R}$. We immediately deduce that, also,

$$|a| - |b| \le |a - b|.$$

Observation 2.1. *Let* f *and* g *be functions continuous at* x_0*. Then also* $f(x) \pm g(x)$ *and* $f(x) \cdot g(x)$ *are continuous at* x_0*. Moreover, if* $g(x_0) \neq 0$ *then* $\frac{f(x)}{g(x)}$ *will be continuous at* x_0*.*

Proof. Proof: We prove it for $f + g$ as $f - g$ can be done similarly. Due to continuity we have $\forall \varepsilon > 0\ \exists \delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ and $|g(x) - g(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta$. But this means that (due to the triangle inequality)

$$|f(x) + g(x) - (f(x_0) + g(x_0))| < |f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon.$$

Now we turn our attention to the product rule. First of all, since $f(x_0)$ is real and the function is continuous, there exists $\delta_1 > 0$ and $M_1 > 0$ such that $|f(x)| < M_1$ whenever $x \in (x_0 - \delta_1, x_0 + \delta_1) \cap \mathrm{Dom}\ f$ (see exercises at the end of this section). Similarly, there exists $\delta_2 > 0$ and $M_2 > 0$ such that $|g(x)| < M_2$ whenever $x \in (x_0 - \delta_2, x_0 + \delta_2) \cap \mathrm{Dom}\ f$. Due to continuity, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2M_2}$ and $|g(x) - g(x_0)| < \frac{\varepsilon}{2M_1}$ for all $x \in (x_0 - \delta, x_0 + \delta)$. We may moreover assume that $\delta < \min\{\delta_1, \delta_2\}$. Then we have

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\le |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)| < \varepsilon \end{aligned}$$

for all $x \in (x_0 - \delta, x_0 + \delta)$.

To prove the last claim it suffices to show that $\frac{1}{g}$ is continuous at x_0 and to use the just proven product rule. Without loss of generality, assume that $g(x_0) > 0$ and denote its value by $y_0 = g(x_0)$. Then, due to the continuity of g, there exists $\delta_1 > 0$ such that $g(x) > \frac{y_0}{2}$

for all $x \in (x_0 - \delta_1, x_0 + \delta_1) \cap \mathrm{Dom}\ g$. Further, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|g(x) - g(x_0)| < y_0^2 \frac{\varepsilon}{2}$ for each $x \in (x_0 - \delta, x_0 + \delta)$ and, moreover, we assume that $\delta < \delta_1$. Then we have

$$\left|\frac{1}{g(x)} - \frac{1}{g(x_0)}\right| = \left|\frac{g(x_0) - g(x)}{g(x)g(x_0)}\right| \leq \frac{|g(x_0) - g(x)|}{y_0 \frac{y_0}{2}} < \varepsilon$$

for each $x \in (x_0 - \delta, x_0 + \delta) \cap \mathrm{Dom}\ g$. □

It is easy to deduce that $f(x) \equiv c$ and $f(x) = x$ are continuous on $\mathbb{R}$.

Definition 2.5. *We say that f is an* odd function *if*

$$\forall x \in \mathrm{Dom}\ f,\ -x \in \mathrm{Dom}\ f \text{ and } f(-x) = -f(x).$$

We say that f is an even function *if*

$$\forall x \in \mathrm{Dom}\ f,\ -x \in \mathrm{Dom}\ f \text{ and } f(-x) = f(x).$$

2.1 Further comments on continuous functions

This section is devoted to advanced properties of continuous functions. They will be mentioned without a proof which is usually not elementary.

Before that, we introduce a notion of a maximum and minimum of set $A \subset \mathbb{R}$.

Definition 2.6. *Let* $\sup A$ *be an element of $A \subset \mathbb{R}$. Then* $\sup A$ *is the highest number of A (or a maximum of A) and we write* $\sup A = \max A$*. Similarly, if* $\inf A$ *is an element of A, then* $\inf A$ *will be the lowest number of A (or a minimum of A) and we write* $\inf A = \min A$*.*

The minimum and maximum does not necessarily exists for a general set $A \subset \mathbb{R}$. For example, $A = \left\{\frac{1}{n},\ n \in \mathbb{N}\right\}$ has maximum 1, however, minimum does not exists. The infimum 0 is not contained in this set.

Note also that every set $A \subset \mathbb{R}$ with finitely many elements has its maximum and minimum.

Definition 2.7. *Let f be continuous on an interval $I \subset \mathbb{R}$. Then we write $f \in \mathcal{C}(I)$.*

Theorem 2.1 (Weierstrass)**.** *Let $f \in \mathcal{C}([a, b])$. Then f is bounded and there exists $t, u \in [a, b]$ such that $f(u) \leq f(x) \leq f(t)$ for all $x \in [a, b]$.*

Actually, the previous theorem states that every function which is continuous on a closed interval attains its maximum and minimum value.

Theorem 2.2 (Bolzano)**.** *Let $f \in \mathcal{C}([a, b])$ and $f(a)f(b) < 0$. Then there is $\eta \in (a, b)$ such that $f(\eta) = 0$.*

Lemma 2.1. *Let f be an odd function and $(-a, a) \subset \mathrm{Dom}\ f$ for some $a > 0$. Then $f(0) = 0$.*

2.2 Elementary functions

Now we are in position where we can define and state basic properties of functions which will be of use hereinafter.

2.2.1 Polynomials

Polynomials are function which arises from a constant function $f \equiv c$, $c \in \mathbb{R}$ and an identity function $f(x) = x$ by finite number of multiplication and additions. In particular, every polynomial is of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0,$$

where $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$. The numbers $a_0, \ldots, a_n$ are called coefficients. The degree of $p(x)$ is n in a case $a_n \neq 0$ and we write deg $p = n$. The term $a_n x^n$ is called a *leading term.* Recall that $p(x) = x^n$ is odd function for odd n and it is an even function for n even. The maximal domain of $p(x)$ is always $\mathbb{R}$. All x such that $p(x) = 0$ are called *roots* of polynomial p. Let x_0 be a root of $p(x)$. Then $p(x) = (x - x_0)q(x)$ where $q(x)$ is a polynomial and it holds that deg $p(x) =$ deg $q(x) + 1$.

2.2.2 Rational functions

A *rational function* is a fraction whose nominator and denominator are polynomials. I.e., a rational function f is of the form

$$f(x) = \frac{p(x)}{q(x)}.$$

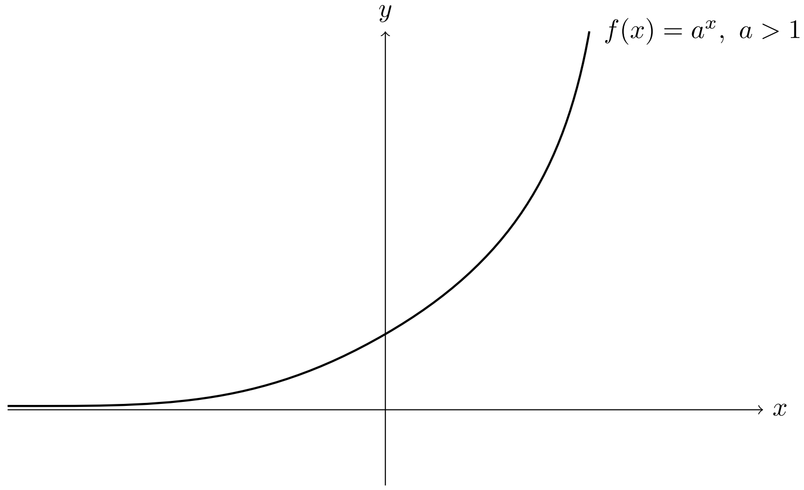
The domain of f is all real numbers except roots of $q(x)$.

2.2.3 Exponential and logarithm

Consider a number $a > 0$. Let $n \in \mathbb{N}$, we define $a^n = a \cdot a \cdot \ldots \cdot a$ where a appears n times on the right hand side. Further, we define $a^{\frac{1}{n}}$ as such number b that $b^n = a$. This allows to define a^r for all rational numbers $r \in Q$. Namely, let $r > 0$, we define $a^r = a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}}$. For $r < 0$ we take $a^r = \frac{1}{a^{-r}}$. Finally, we are allowed to define uniqely a continuous function

$$f(x) = a^x \tag{1}$$

whose values are prescribed in the aforementioned way for all rational inputs. Since the function is constant for $a \equiv 1$, we remove this particular base from our definition and we consider the relation (1) only for $a \in (0, 1) \cup (1, \infty)$. It holds that Dom $f = \mathbb{R}$ and Ran $f = (0, \infty)$. Further, $f(0) = 1$ (roughly speaking, every number powered to 0 equals one). The function is strictly increasing for $a > 1$ and strictly decreasing for $a < 1$. The picture below is a graph of a function $f(x) = a^x$ for some $a > 0$.

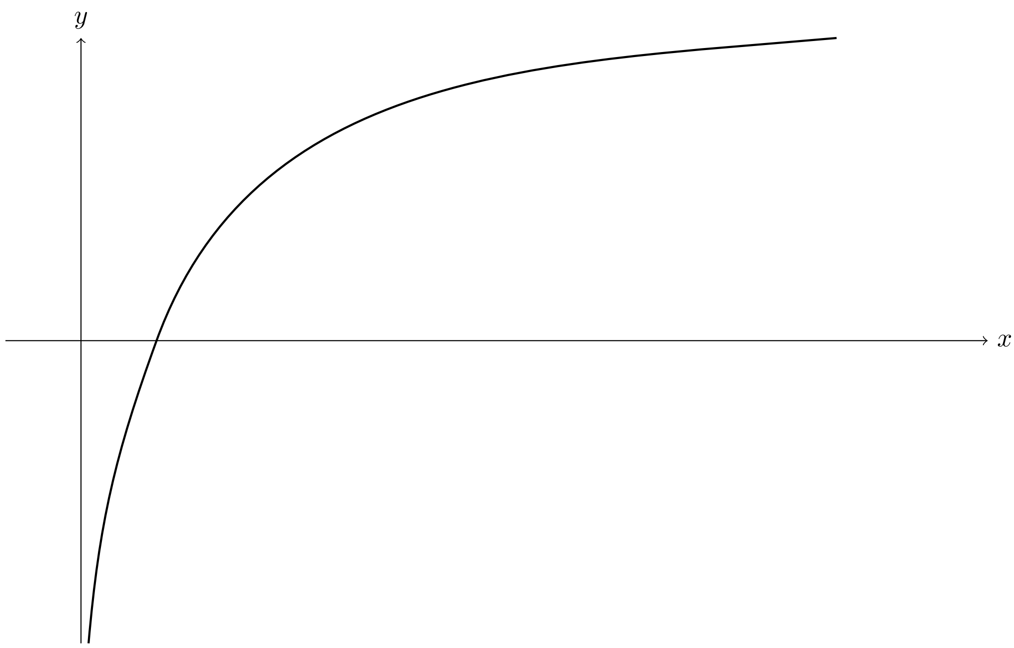


Since $x \mapsto a^x$ is injective there exists an inverse function. We will denote it by $\log_a$ and it is called logarithm to base a. In particular

$$\log_a y = x \quad \Leftrightarrow \quad a^x = y.$$

Recall that $a \in (0,1) \cup (1,\infty)$ and, due to the properties of the inverse functions, Dom $\log_a = (0,\infty)$ and Ran $\log_a = \mathbb{R}$. Recall also, that since $a^0 = 1$, we have $\log_a 1 = 0$ for every $a \in (0,1) \cup (1,\infty)$.

The graph of $f(x) = \log_a(x)$, $a > 1$ is the following



Let e be Euler's number (for its definition see relation (3)). The logarithm to base e is called *natural logarithm* and, because of its importance, we omit the index e in its notation (i.e. $\log x = \log_e x$).

2.2.4 Irrational functions

Next, we define *nth root* $f(x) = \sqrt[n]{x}$ as an inverse to $g(x) = x^n$. Recall that g is invertible for n odd and $\mathrm{Dom}\ g = \mathrm{Ran}\ g = \mathbb{R}$. Thus, $\mathrm{Dom}\ \sqrt[n]{x} = \mathrm{Ran}\ \sqrt[n]{x} = \mathbb{R}$ for n odd.

However, g is not invertible for n even. In that case we have to restrict the domain of g to $[0,\infty)$ in order to have an injective function. The range of this restricted function is also $[0,\infty)$. As a consequence, $\mathrm{Dom}\ \sqrt[n]{x} = \mathrm{Ran}\ \sqrt[n]{x} = [0,\infty)$ for n even.

The nth root is always an increasing function.

2.2.5 Trigonometric functions

There is just one pair of continuous functions $s(x)$ and $c(x)$ with the following properties

- $s(x)^2 + c(x)^2 = 1$
- $s(x+y) = s(x)c(y) + c(x)s(y)$
- $c(x+y) = c(x)c(y) - s(x)s(y)$
- $0 < xc(x) < s(x) < x$ for all $x \in (0,1)$.

The function s is called sinus and the function c is called cosine. We also introduce notation $\sin x = s(x)$ and $\cos x = c(x)$. These functions have the following properties:

- $\mathrm{Dom}\ \sin x = \mathrm{Dom}\ \cos x = \mathbb{R}$, $\mathrm{Ran}\ \sin x = \mathrm{Ran}\ \cos x = [-1,1]$.
- $\sin x$ is an odd function, $\cos x$ is an even function.
- $\sin x$ and $\cos x$ are 2π periodic function.

There are several 'known' values of sin and cos:

$x=$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0

Besides, we define a function $\tan x = \frac{\sin x}{\cos x}$ (tangens) and a function $\cot x = \frac{\cos x}{\sin x}$ (cotangens). These functions are $\pi-$periodic, their range is $\mathbb{R}$ and

$$\mathrm{Dom}\ \tan x = \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\},\ \mathrm{Dom}\ \cot x = \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}.$$

2.2.6 Cyclometric functions

Roughly speaking, cyclometric functions are inverse functions to the aforementioned trigonometric functions. However, every trigonometric function is periodic and thus it is not one-to-one. To obtain the inverse function, we have to restrict the domain of every trigonometric function. In particular, we define functions $\sin_r$, $\cos_r$, $\tan_r$ and $\cot_r$ as follows

$$\begin{aligned}
\sin_r x &= \sin x,\ \mathrm{Dom}\ \sin_r = [-\pi_2, \pi_2] \\
\cos_r x &= \cos x,\ \mathrm{Dom}\ \cos_r = [0, \pi] \\
\tan_r x &= \tan x,\ \mathrm{Dom}\ \tan_r = [-\pi_2, \pi_2] \\
\cot_r x &= \cot x,\ \mathrm{Dom}\ \cot_r = [0, \pi]
\end{aligned}$$

Now, since these functions are injective, we may define

$$\begin{aligned}\arcsin &= \sin_r^{-1}\\ \arccos &= \cos_r^{-1}\\ \arctan &= \tan_r^{-1}\\ \operatorname{arccot} &= \cot_r^{-1}\end{aligned}$$

Let write down several properties of each function:

- $\mathrm{Dom}\ \arcsin = [-1,1]$, $\mathrm{Ran}\ \arcsin = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, arcsin is an increasing function and $\arcsin(-1) = -\frac{\pi}{2}$, $\arcsin(0) = 0$ and $\arcsin(1) = \frac{\pi}{2}$
- $\mathrm{Dom}\ \arccos = [-1,1]$, $\mathrm{Ran}\ \arccos = [0,\pi]$, arccos is a decreasing function and $\arccos(-1) = \pi$, $\arccos(0) = \frac{\pi}{2}$ and $\arccos(1) = 0$.
- $\mathrm{Dom}\ \arctan = \mathbb{R}$, $\mathrm{Ran}\ \arctan = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, arctan is an increasing function and $\arctan(0) = 0$.
- $\mathrm{Dom}\ \operatorname{arccot} = \mathbb{R}$, $\mathrm{Ran}\ \operatorname{arccot} = (0,\pi)$, arccot is a decreasing function and $\operatorname{arccot}(0) = \frac{\pi}{2}$.

2.3 Exercises

1. Try to think about the following statement: If both functions $f(x)$ and $g(x)$ are not monotone on $\mathbb{R}$, then their sum $f(x)+g(x)$ is not monotone on $\mathbb{R}$.

   Prove if it is true, find a counterexample if it is false.
2. If a function is not monotone, then it does not have an inverse function. It is true or false? And why?
3. Let f be increasing invertible function. Show that f^{-1} is also increasing. Consider also the case of decreasing invertible function.
4. Use a definition of continuity in order to proof that a function $f(x) = x^2\chi_{(-1,1)} + \chi_{[1,3]}$ is continuous in $x_0 = 1$.
5. Determine all points of continuity of a function $f(x) = x\chi_{\mathbb{Q}} - x\chi_{\mathbb{R}\setminus\mathbb{Q}}$.
6. Find all roots of $p(x) = x^3 - 6x^2 + 11x - 6$.
7. Let f be continuous at x_0. Then there exists $\delta > 0$ and $M > 0$ such that $|f(x)| < M$ for every $x \in (x_0-\delta, x_0+\delta) \cap \mathrm{Dom}\, f$. Prove or disprove this claim.
8. Deduce the values of $\sin x$ and $\cos x$ for $x = \frac{2}{3}\pi,\ \frac{3}{4}\pi,\ \frac{5}{6}\pi,\ \frac{7}{6}\pi,\ \frac{5}{4}\pi,\ \frac{4}{3}\pi,\ \frac{5}{3}\pi,\ \frac{7}{4}\pi,\ \frac{11}{6}\pi$.

3 Sequences and their limits, introduction

Definition 3.1. *A function* $a : \mathbb{N} \mapsto \mathbb{R}$, $\mathrm{Dom}\ a = \mathbb{N}$ *is called* sequence. We write a_n instead of $a(n)$. The whole function is then denoted $\{a_n\}_{n=1}^{\infty}$.

For example, $a_n = \frac{1}{n}$ is a sequence of numbers $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Sequence $b_n = 2^n$ is a sequence of numbers $\{2, 4, 8, 16, 32, \ldots\}$. Note also that the first sequence can be written as $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and the second one as $\{2^n\}$.

Note that the sequence is actually a real function as considered in the previous chapter whose domain is a set of natural numbers. Thus, one can talk about boundedness and monotony in means of Definitions 1.9 and 2.2. Nevertheless, let recall a definition of a monotonous sequence which is more convenient for use then Definition 2.2.

Definition 3.2. *A sequence a_n is called*

- increasing, if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$,
- *decreasing, if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$,*
- non-increasing, if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$,
- *non-decreasing, if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$.*

A sequence, which posses one of these properties is monotonous.

Definition 3.3. *Let a_n be a sequence. A number $A \in \mathbb{R}$ is called a limit of a_n if*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n > n_0,\ |a_n - A| < \varepsilon.$$

We then write $\lim a_n = A$.
A limit of a_n is $+\infty$ if

$$\forall M > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n > n_0,\ a_n > M$$

and we write $\lim a_n = +\infty$.
A limit of a_n is $-\infty$ if $\lim -a_n = +\infty$.

Observation 3.1. *Let a_n be a sequence and let $A \in \mathbb{R}^*$ be its limit. Then it is determined uniquely.*

Proof. Let there be two numbers $A, B \in \mathbb{R}$, $A \neq B$ (here we assume, for simplicity, that both numbers are real, for other cases see exercises) and let $\lim a_n = A$ and $\lim b_n = B$. Take $\varepsilon = \frac{1}{3}|A - B|$. According to definition, there exists n_0 such that $|a_n - A| < \varepsilon$ for all $n > n_0$ and there exists n_1 such that $|a_n - B| < \varepsilon$ for all $n > n_1$. Take $n > \max\{n_0, n_1\}$. Then

$$|A - B| = |A - a_n + a_n - B| \leq |A - a_n| + |B - a_n| < \varepsilon + \varepsilon < 3\varepsilon = |A - B|$$

which is of course a contradiction. □

Consider a sequence $a_n = \frac{1}{n}$. We claim that $\lim a_n = 0$. Indeed, let $\varepsilon > 0$ be an arbitrary number. Take $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\varepsilon}$. Then for all $n > n_0$ we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{n_0} < \varepsilon.$$

Next, consider a sequence $a_n = n$ (i.e. a sequence $\{1, 2, 3, \ldots\}$). We claim that $\lim a_n = \infty$. To prove this, let $M > 0$ be an arbitrary number. Take a natural number n_0 such that $n_0 > M$. Then for all $n > n_0$ we have $a_n = n > n_0 > M$.

Lemma 3.1 (Arithmetic of limits)**.** *Let a_n and b_n be sequences and let $c \in \mathbb{R}$. Then*

$$\begin{aligned}
\lim\,(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\
\lim\,(a_n b_n) &= \lim a_n \cdot \lim b_n \\
\lim c a_n &= c \lim a_n \\
\lim \frac{a_n}{b_n} &= \frac{\lim a_n}{\lim b_n}
\end{aligned}$$

assuming the right hand side has meaning.

To make the lemma complete we specify what is the 'meaning of the right hand side'. Besides the usual division by zero there are several others indefinite terms

$$\infty - \infty,\ \frac{\infty}{\infty},\ 0 \cdot \infty,\ \frac{0}{0},\ 1^{\infty},\ \infty^0,\ 0^0$$

which do not have any meaning. We also recall that $\frac{1}{\infty} = 0$.

Proof. Here we proof only a simplified version of this claim as we will assume that $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$. Further, as the proof does not differ from the one of Observation 2.1, we consider only $\lim(a_n + b_n) = \lim a_n + \lim b_n$. Take $\varepsilon > 0$ arbitrarily. Since $\lim a_n = A$ and $\lim b_n = B$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - A| < \frac{1}{2}\varepsilon$ and $|b_n - B| < \frac{1}{2}\varepsilon$. Consequently,

$$|a_n + b_n - A - B| \le |a_n - A| + |b_n - B| < \varepsilon$$

and we have just verified that $A + B$ is a limit of $a_n + b_n$. $\square$

Let compute several limits. First of all, we will prove that $\lim q^n = \infty$ for $q > 1$. Thus, we have to show that for every M there is n_0 such that $q^n > M$. In this case, it is enough to take such natural number n_0 that $n_0 > \log_q M$. Then, necessarily, $q^n > q^{n_0} > q^{\log_q M} > M$ since $f(x) = q^x$ is increasing.

Next, we compute $\lim n^2 - n$. One may try to write $\lim n^2 - n = \lim n^2 - \lim n = \infty - \infty$. However, the last term is an indefinite term and the arithmetic of limit cannot be used in such way. We will proceed as follows

$$\lim n^2 - n = \lim n^2 \left(1 - \frac{1}{n}\right) = \lim n^2 \left(1 - \lim \frac{1}{n}\right) = \infty(1 - 0) = \infty.$$

The general rule how to compute a limit of 'rational sequence' is to divide by the highest power of n appearing in the denominator. Let demonstrate this (in both cases we use the arithmetic of limits):

$$\begin{aligned}
\lim \frac{n+1}{n^2+3} &= \lim \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{3}{n^2}} = \frac{0+0}{1+0} = 0, \\
\lim \frac{n^3 + 3n^2}{3n^3 + n^2} &= \lim \frac{1 + 3\frac{1}{n}}{3 + \frac{1}{n}} = \frac{1 + 3 \cdot 0}{3 + 0} = \frac{1}{3}.
\end{aligned}$$

Let compute a limit $\lim q^n$ with $q \in (0,1)$. By use of the arithmetic of limits and the previous claim we compute

$$\lim q^n = \lim \left(\frac{1}{\frac{1}{q}}\right)^n = \frac{1}{\lim \left(\frac{1}{q}\right)^n} = \frac{1}{\infty} = 0.$$

Before the next observation let us recall that the notion of 'bounded sequence' was already defined. It follows from the definition of sequence (Definition 3.1 – in particular it is a function whose domain is $\mathbb{N}$) and from the definition of a bounded function (Definition 1.9).

Observation 3.2. *Let a_n be a sequence with real (finite) limit A. Then a_n is a bounded sequence.*

Proof. Indeed, take (for instance) $\varepsilon = 1$. There exists $n_0 \in \mathbb{N}$ such that $\{a_n\}_{n>n_0}$ is bounded from above by $A+1$ and from below by $A-1$. Next, $\{a_1, a_2, \ldots, a_{n_0}\}$ is a finite set and thus it is bounded from above (say by $M \in \mathbb{R}$) and from below by $m \in \mathbb{R}$. Then, $\{a_n\}_{n=1}^{\infty}$ is bounded from above by $\max\{M, A+1\}$ and from below by $\min\{m, A-1\}$. $\square$

Lemma 3.2 (Sandwich lemma)**.** *Let a_n, b_n, c_n be such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Assume, moreover, that* $\lim a_n = \lim c_n = A \in \mathbb{R}^*$. *Then* $\lim b_n$ *exists and* $\lim b_n = A$.

Proof. Take an arbitrary $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $|a_n - A| < \varepsilon$ and $|c_n - A| < \varepsilon$. There may appear one of the following cases:

- $A \geq c_n$. In that case, $|b_n - A| \leq |a_n - A| < \varepsilon$.
- $A \leq a_n$. In that case, $|b_n - A| \leq |c_n - A| < \varepsilon$.
- $A \in (a_n, c_n)$. In that case, since $b_n \in [a_n, c_n]$, we have $|b_n - A| < |a_n - c_n| = |a_n - A + A - c_n| \leq |a_n - A| + |b_n - B| < 2\varepsilon$.

No matter which one is true, we have $|b_n - A| < 2\varepsilon$ and A is a limit of b_n according to the definition of limit. $\square$

Definition 3.4. *Let a_n be a sequence and let $k : \mathbb{N} \mapsto \mathbb{N}$ be an increasing sequence of natural numbers. Then a_{k_n} is a* subsequence.

Observation 3.3. *Let a_n be a sequence such that* $\lim a_n = A$, $A \in \mathbb{R}^*$. *Then every subsequence a_{k_n} has a limit A.*

Proof. Once again, we assume for simplicity that $A \in \mathbb{R}$. For arbitrary $\varepsilon > 0$ there exists n_0 such that $|a_n - A| < \varepsilon$. However, as k_n is an increasing sequence of natural numbers, there exists $n_1 \in \mathbb{N}$ such that $k_n > n_0$ whenever $n > n_1$. That means that for ever $n > n_1$ we have $|a_{k_n} - A| < \varepsilon$. The proof is complete. $\square$

4 Limits of functions

4.1 Limits

Definition 4.1. *A* limit point *of a set $S \subset \mathbb{R}$ is every point $x_0 \in \mathbb{R}$ such that for every $\delta > 0$ it holds that $((x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \cap S \neq \emptyset$.*

Consider, for example, $S = (0,1) \cup \{2\}$. The set of all its limit point is a closed interval $[0,1]$. We are ready to define a limit of a function. First, we consider finite limits.

Definition 4.2. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ and let x_0 be a limit point of* Dom f. *We say, that $A \in \mathbb{R}$ is a* limit of f at x_0 *if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in ((x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \cap \mathrm{Dom}\ f,\ \ |f(x) - A| < \varepsilon.$$

We write

$$\lim_{x \to x_0} f(x) = A$$

Observation 4.1. *Once the limit exists, it is determined uniquely.*

Proof. Let $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} f(x) = B$ for some different $A, B \in \mathbb{R}$. Take $\varepsilon = \frac{1}{3}|B-A|$. According to the definition of a limit, there exists $\delta > 0$ such that $|f(x)-A| < \varepsilon$ and, simultaneously, $|f(x)-B| < \varepsilon$ for some $x \in (x_0-\delta, x_0+\delta)$. We use the triangle inequality to deduce

$$|A-B| = |A-f(x)+f(x)-B| \leq |A-f(x)| + |f(x)-B| \leq \frac{2}{3}|A-B|.$$

□

Thus, the definition of the limit is correct.

Observation 4.2. *Let f be a function continuous in a limit point x_0 of* Dom f *. Then*

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Proof. Let $\varepsilon > 0$ be arbitrary. As f is continuous, there exists $\delta > 0$ such that $|f(x)-f(x_0)| < \varepsilon$ whenever $|x-x_0| < \varepsilon,\ x \in \text{Dom } f$. But that is exactly that δ which suits the definition of a limit. □

Here we would like to emphasize that every elementary function from the previous chapter is continuous on its domain.

This is the first tool which allows a computation. For example

$$\lim_{x\to 3} x - 5 = -2.$$

Ok, that was too easy. Anyway, we may use it to simplify fractions. Consider for example a function $f(x) = \frac{x^2+4x+3}{x^2-1}$. This function is clearly not defined at points -1 and 1 and is continuous everywhere else. Anyway, we may compute

$$\lim_{x\to -1} \frac{x^2+4x+3}{x^2-1} = \lim_{x\to -1} \frac{(x+1)(x+3)}{(x-1)(x+1)} = \lim_{x\to -1} \frac{x+3}{x-1} = -1$$

Definition 4.3. *Let x_0 be a limit point of* Dom f*. We say that $A \in \mathbb{R}$ is a* left-sided limit *of f at x_0 (resp.* right-sided limit *of f in x_0) if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (x_0-\delta, x_0) \cap \text{Dom } f,\ |f(x)-A| < \varepsilon.$$

(resp.

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (x_0, x_0+\delta) \cap \text{Dom } f,\ |f(x)-A| < \varepsilon.)$$

We write

$$\lim_{x\to x_0-} f(x) = A \quad (\textit{resp.} \lim_{x\to x_0+} f(x) = A).$$

A special case of the one-sided limit is a limit at infinity. This is defined as follows

Definition 4.4. *Let for all $K \in \mathbb{R}$ there be $x \in$ Dom f such that $x > K$. We say that $A \in \mathbb{R}$ is a limit of f at ∞ if*

$$\forall \varepsilon > 0,\ \exists K \in \mathbb{R},\ \forall x > K,\ x \in \text{Dom } f,\ |f(x)-A| < \varepsilon.$$

We write $\lim_{x\to\infty} f(x) = A$.
We say that A is a limit of $f(x)$ at $-\infty$ if A is a limit of $f(-x)$ at ∞. We write $\lim_{x\to-\infty} f(x) = A$.

Besides that, we define also infinite limits

Definition 4.5. *Let x_0 be a limit point of* Dom f. *We say that $+\infty$ is a limit of f at a point x_0 if*

$$\forall K > 0,\ \delta > 0,\ \forall x \in ((x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \cap \mathrm{Dom}\ f,\ f(x) > K.$$

We write $\lim_{x\to x_0} f(x) = +\infty$.
We say that $-\infty$ is a limit of f at x_0 if $\lim_{x\to x_0} -f(x) = +\infty$. *We write* $\lim_{x\to x_0} f(x) = -\infty$.

Of course, one can define also one-sided infinite limits and infinite limits in infinity. We left it to reader as an exercise.

The following observation is one of the most crucial tool for the computation of limits. We call it 'arithmetic of limits'.

Lemma 4.1 (Arithmetic of limits)**.** *Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ and let x_0 be a limit point of* Dom f *and* Dom g. *Let, moreover, $c \in \mathbb{R}$. Then*

$$\begin{aligned}
\lim_{x\to x_0} (f(x) \pm g(x)) &= \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) \\
\lim_{x\to x_0} cf(x) &= c \lim_{x\to x_0} f(x) \\
\lim_{x\to x_0} (f(x)g(x)) &= \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x) \\
\lim_{x\to x_0} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}
\end{aligned} \tag{2}$$

assuming the right hand side has meaning.

Let recall that we compute with infinity once again. So this time we have the same indefinite terms as previously, see Lemma 3.1.

We postpone the proof of this lemma to the next section.

Note that the arithmetic of limits holds also for the one-sided limits.

Let compute a limit $\lim_{x\to\infty} \frac{x-1}{x-2}$. According to arithmetic of limits $\lim_{x\to\infty} x - 1 = \infty$ and $\lim_{x\to\infty} x - 2 = \infty$. However, we cannot write that

$$\lim_{x\to\infty} \frac{x-1}{x-2} = \frac{\infty}{\infty}$$

as we get an indefinite term. The solution makes use of $\lim_{x\to\infty} \frac{1}{x} = 0$. This particular limit is left as an exercise.

$$\lim_{x\to\infty} \frac{x-1}{x-2} = \lim_{x\to\infty} \frac{1-\frac{1}{x}}{1-\frac{2}{x}} = \frac{1-0}{1-2\cdot 0} = 1$$

where we first multiply the numerator and denominator by $\frac{1}{x}$ and, second, we use the arithmetic of limits.

Observation 4.3. *Let* $\lim_{x\to x_0} f(x) = A$ *for some* $x_0 \in \mathbb{R}$ *and* $A \in \mathbb{R}^*$. *Then also* $\lim_{x\to x_0-} f(x) = A$ *and* $\lim_{x\to x_0+} f(x) = A$.

Once again, the proof of this observation is postponed to the next section.

Let consider $\lim_{x\to 0} \frac{1}{x}$. We are going to show that $\lim_{x\to 0-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0+} \frac{1}{x} = +\infty$. In such case, $\lim_{x\to 0} \frac{1}{x}$ does not exist according to the just mentioned observation.

Let $K > 0$. We take $\delta = \frac{1}{K}$ and, consequently, for all $x \in (0, \delta)$ it holds that $f(x) = \frac{1}{x} > \frac{1}{\delta} = K$ and $\lim_{x\to 0+} \frac{1}{x} = \infty$.

Similarly, for all $x \in (-\delta, 0)$ it holds that $f(x) = \frac{1}{x} < \frac{1}{\delta} = -K$ and thus $\lim_{x\to 0-} \frac{1}{x} = -\infty$.

4.2 Advanced limits

There is precisely one real number e such that

$$\lim_{x\to 0}\frac{e^x-1}{x}=1. \tag{3}$$

This number is called Euler's number, it is irrational and its value is approximately 2.72.

Thus we also get

$$\lim_{x\to 0}\frac{\log(x+1)}{x}=1$$

The definition of $\sin x$ allows to deduce

$$\lim_{x\to 0}\frac{\sin x}{x}=0. \tag{4}$$

Lemma 4.2 (Limit of composed function)**.** *Let* $\lim_{x\to x_0} g(x)=A$ *and* $\lim_{y\to A} f(y)=B$. *Then*

$$\lim_{x\to x_0} f(g(x))=B,$$

if at least one of the following is true:

1. *f is continuous at the point A or*
2. *there is δ such that for all $x\in(x-\delta,x_0)\cup(x_0,x+\delta)$ it holds that $g(x)\neq A$.*

Now we are allowed to deduce further limits which will be used without any further explanation (here note that the inner function $g(x)=\frac{x}{2}$ is injective and thus the second assumption of the previous lemma is fulfilled):

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}{x^2}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{4\left(\frac{x}{2}\right)^2}$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}=\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}=\frac{1}{2}$$

Lemma 4.3 (Heine)**.** *Let $f:\mathbb{R}\mapsto\mathbb{R}$. Then $\lim_{x\to x_0} f(x)=A$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$, $x_n\neq x_0$ for all $n\in\mathbb{N}$ it holds that*

$$\lim x_n=x_0\Rightarrow \lim f(x_n)=A.$$

Lemma 4.4 (Sandwich Lemma)**.** *Let $x_0\in\mathbb{R}$ and let there is $\delta>0$ such that*

$$f(x)\le g(x)\le h(x),\ \forall x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta).$$

Then $\lim_{x\to x_0} f(x)=\lim_{x\to x_0} h(x)=A$ implies $\lim_{x\to x_0} g(x)=A$.

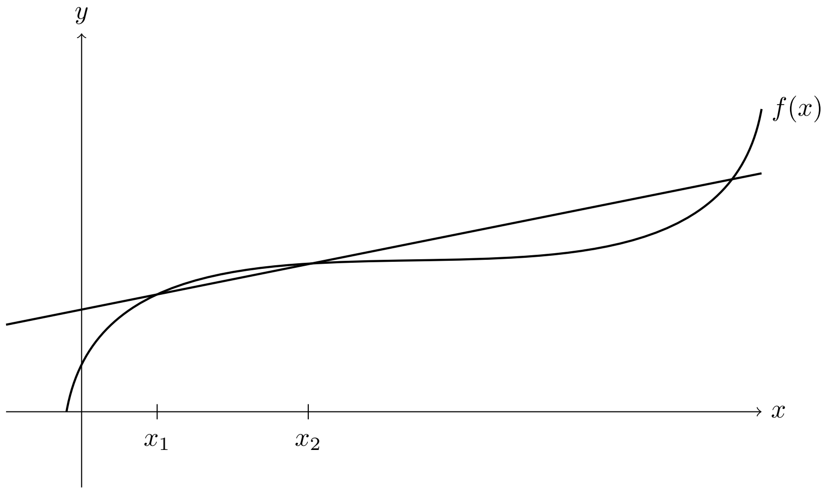
Further limits of elementary functions:

- $\lim_{x\to\infty} a^x=\infty$ for $a>1$,
- $\lim_{x\to\infty}\log_a x=\infty$ for $a>1$,

- $\lim_{x\to 0+} \log_a x = -\infty$ for $a > 1$,
- $\lim_{x\to \frac{\pi}{2}-} \tan x = \infty$,
- $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$,
- $\lim_{x\to\infty} \operatorname{arccot} x = 0$,
- $\lim_{x\to-\infty} \operatorname{arccot} x = \pi$.

4.3 Derivative

Consider a graph of a function $f(x)$, for example, of the following form



The equation of the line passing through point $\langle x_1, f(x_1)\rangle$ and $\langle x_2, f(x_2)\rangle$ is

$$y = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1).$$

How to make a tangent line? Just simply tend with x_2 to x_1. So the tangent line has equation

$$y = k(x - x_1) + f(x_1)$$

where

$$k = \lim_{x_2\to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

assuming the limit exists. We denote $h := x_2 - x_1$ and then we may write

$$k = \lim_{h\to 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

Definition 4.6. *Let $f : \mathbb{R} \mapsto \mathbb{R}$. We define*

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}.$$

We say that $f'(x)$ is a derivative *of f at point x.*

In particular, a derivative of f in a point x is a slope of the tangent line passing through $\langle x, f(x)\rangle$.

Let emphasize that f' does not exist for every function.

Observation 4.4. *Let $f'(x_0)$ is real. Then f is continuous at x_0.*

Proof. Indeed, it is enough to compute

$$\lim_{x\to x_0} f(x) - f(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = f'(x_0)\cdot 0 = 0.$$

Consequently, $\lim_{x\to x_0} f(x) = f(x_0)$ and the function is continuous at x_0. □

Let compute several derivatives of elementary functions. First of all, since

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}),$$

we get for $f(x) = x^n$

$$\begin{aligned}\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} &= \lim_{h\to 0}\frac{h((x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1})}{h}\\ &= \lim_{h\to 0}(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1} = nx^{n-1}.\end{aligned}$$

Thus, $(x^n)' = nx^{n-1}$.

Take $f(x) = e^x$.

$$\lim_{h\to 0}\frac{e^{x+h} - e^x}{h} - \lim_{h\to 0}\frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0}\frac{e^h - 1}{h} = e^x.$$

Consequently, $(e^x)' = e^x$. Consider $f(x) = \sin x$. We compute

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} &= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h}\\ &= \lim_{h\to 0}\left(\frac{\sin h\cos x}{h} - \frac{\sin x(1-\cos h)}{h}\right) \overset{AL}{=} \cos x\lim_{h\to 0}\frac{\sin h}{h} - \sin x\lim_{h\to 0}\frac{1-\cos h}{h^2}h \overset{AL}{=} \cos x\end{aligned}$$

and we deduced that $(\sin x)' = \cos x$.

Let compute derivative of $f(x) = \cos x$:

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} &= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\\ &= \lim_{h\to 0}\left(\frac{\cos x(\cos h - 1)}{h} + \frac{-\sin x\sin h}{h}\right).\end{aligned}$$

Similarly as before we deduce that

$$(\cos x)' = -\sin x.$$

Finally, let compute a derivative of $\log x$. We have

$$\lim_{h\to 0}\frac{\log(x+h)-\log x}{h}=\lim_{h\to 0}\frac{\log\left(\frac{x+h}{x}\right)}{h}=\lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{h}=\lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\frac{1}{x}\overset{LOCF}{=}\frac{1}{x}.$$

Consequently,

$$(\log x)'=\frac{1}{x}.$$

Lemma 4.5. *Let f and g be differentiable functions. Then*

$$\begin{aligned}(f(x)\pm g(x))'&=f'(x)\pm g'(x)\\(f(x)g(x))'&=f'(x)g(x)+f(x)g'(x)\\\left(\frac{f(x)}{g(x)}\right)'&=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}\end{aligned}$$

if both sides have sense.

Proof. The first relation follows directly from the arithmetic of limits. Indeed,

$$\begin{aligned}(f(x)\pm g(x))'&=\lim_{h\to 0}\frac{f(x+h)\pm g(x+h)-(f(x)\pm g(x))}{h}\\&=\lim_{h\to 0}\frac{(f(x+h)-f(x))\pm(g(x+h)-g(x))}{h}\\&\overset{AL}{=}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\pm\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f'(x)\pm g'(x).\end{aligned}$$

Further,

$$\begin{aligned}(f(x)g(x))'&=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\&=\lim_{h\to 0}\frac{f(x+h)(g(x+h)-g(x))+g(x)(f(x+h)-f(x)}{h}\\&\overset{AL}{=}\lim_{h\to 0}\frac{f(x+h)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h}\\&=f(x)g'(x)+f'(x)g(x)\end{aligned}$$

which is a proof of the second relation.

Finally,

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)'&=\lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)g(x)-g(x+h)f(x)}{g(x+h)g(x)}\right)\\&=\lim_{h\to 0}\frac{1}{g(x+h)g(x)}\frac{(f(x+h)-f(x))\,g(x)-f(x)(g(x+h)-g(x))}{h}\\&=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}\end{aligned}$$

which proves the last relation. □

Let compute
$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
and, similarly, we may deduce $(\cot x)' = -\frac{1}{\sin^2 x}$.

We present the following lemma without a proof. It concern the derivative of composed functions.

Lemma 4.6. *Let f and g be differentiable functions and let $b = f(a)$. Then*
$$(g \circ f)'(a) = g'(b)f'(a) = g'(f(a))f'(a).$$

So we may use this to compute the derivative of a^x:
$$(a^x)' = \left(e^{x \log a}\right)' = e^{x \log a}(x \log a)' = \log a \ e^{x \log a} = \log a \ a^x.$$

Finally, we may also compute remaining derivatives of elementary functions:
$$1 = (x)' = (\arctan \circ \tan x)' = \arctan'(\tan x)\tan' x.$$
We thus deduce that $\arctan'(\tan x) = \frac{1}{\tan'(x)}$ and thus
$$\arctan'(\tan x) = \cos^2 x = \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{1}{1 + \frac{\sin^2 x}{\cos^2 x}} = \frac{1}{1 + \tan^2 x}$$
which yield
$$\arctan'(x) = \frac{1}{1+x^2}.$$
The similar computation may be performed also for other cyclometric functions. To sum up, we present the following table:

$f(x)$	$f'(x)$	conditions
x^n	nx^{n-1}	$n \in \mathbb{R}$, x as usual
e^x	e^x	$x \in \mathbb{R}$
a^x	$\log a \ a^x$	$a \in (0,1) \cup (1,\infty)$, $x \in \mathbb{R}$
$\log x$	$\frac{1}{x}$	$x \in (0,\infty)$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$
$\cot x$	$-\frac{1}{\sin^2 x}$	$x \in \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$	$x \in \mathbb{R}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$x \in (-1,1)$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$x \in (-1,1)$

4.3.1 Mean-value theorems

Lemma 4.7. *Let f be defined on an interval (a,b) let it attain its maximum (resp. minimum) in a point $x_0 \in (a,b)$, and let $f'(x_0)$ exist. Then $f'(x_0) = 0$.*

Proof. Let x_0 be a point of maximum. For contradiction let $f'(x_0) \neq 0$ and without loss of generality assume $f'(x_0) > 0$. But then there is $\delta > 0$ such that $\frac{f(x)-f(x_0)}{x-x_0} > 0$ for all $x \in (x_0, x_0 + \delta)$. But this means that $f(x) > f(x_0)$ which is in contradiction with the very first assumption. $\square$

Lemma 4.8 (Rolle)**.** *Let $f \in \mathcal{C}([a,b])$ and let f' exist for all $x \in (a,b)$. Moreover, let $f(a) = f(b)$. Then there exists a point $\zeta \in (a,b)$ such that $f'(\zeta) = 0$.*

Proof. For f constant it is enough to take any $x \in (a,b)$. Once f is not constant, there is a point $\zeta \in (a,b)$ where this function attains its maximum or minimum. According to the previous lemma, $f'(\zeta) = 0$. $\square$

Lemma 4.9 (Lagrange)**.** *Let $f \in \mathcal{C}([a,b])$ and let f' exists for all $x \in (a,b)$. Then there exists a point $\zeta \in (a,b)$ such that*

$$f'(\zeta)(b-a) = f(b) - f(a)$$

Proof. Consider a function $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$. This function satisfies all assumption of the previous lemma ($F(a) = F(b) = 0$) and thus there is ζ such that $F'(\zeta) = 0$. This might be rewritten as

$$0 = f'(\zeta) - \frac{f(b) - f(a)}{b-a}$$

which is the desired equality. $\square$

4.3.2 The course of function

The derivative helps to further analyze the function. This is the main content of this section. First of all, the sign of derivative is in correspondence with the monotonicity of function.

Observation 4.5. *Let $f \in \mathcal{C}(I)$ for some interval $I \subset \mathbb{R}$. Assume that $f'(x)$ is exists for all $x \in I$.*

1. *If $f'(x) > 0$ for all $x \in I$, $f(x)$ is increasing on I.*
2. *If $f'(x) < 0$ for all $x \in I$, $f(x)$ is decreasing on I.*

Proof. We prove just the first part as the second part is just an easy modification. Let $x, y \in I$ be arbitrary points such that $x < y$. According to mean value theorem, there is $\zeta \in (x,y)$ such that $f'(\zeta)(x-y) = f(x) - f(y)$. As $f'(\zeta)$ is positive we get $f(x) < f(y)$ which implies the desired claim. $\square$

Definition 4.7. *We say that $x_0 \in \mathrm{Dom}\ f$ is a local maximum of f if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for every $x \in (x_0 - \delta, x_0 + \delta)$. It is a local minimum of f if there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ for every $x \in (x_0 - \delta, x_0 + \delta)$.*

We define one additional qualitative property of function:

Definition 4.8. *We say that $f : \mathbb{R} \mapsto \mathbb{R}$ is convex on a set $I \subset \mathrm{Dom}\ f$ if for all $x, y, z \in I$, $x < y < z$ it holds that*

$$\frac{f(y) - f(x)}{y - x} < \frac{f(z) - f(y)}{z - y}.$$

We say that f is concave on I if $-f$ is convex on I.

Observation 4.6. *Let $f \in \mathcal{C}(I)$ for some interval $I \subset \mathbb{R}$. Assume that $f''(x)$ exists for all $x \in I$.*

1. If $f''(x) > 0$ for all $x \in I$ then f is convex on I.

2. If $f''(x) < 0$ for all $x \in I$ then f is concave on I.

Proof. It is enough to show that f' increasing implies f convex as then the claim follows from Observation 4.5. Take $x, y, z \in I$, $x < y < z$. According to mean value theorem there exist $\eta \in (x, y)$ and $\zeta \in (y, z)$ such that $f'(\eta) = \frac{f(y)-f(x)}{y-x}$ and $f'(\zeta) = \frac{f(z)-f(y)}{z-y}$. But since f' is increasing and $\eta < \zeta$ we get $f'(\eta) < f'(\zeta)$ which implies the convexity of f.

If $f''(x) < 0$ we get $(-f)''(x) > 0$ and according to the first part $-f$ is convex. This gives the second claim. □

Definition 4.9. *We say that $x \in \mathbb{R}$ is a* point of inflection *of $f : \mathbb{R} \mapsto \mathbb{R}$ if f is continuous at x and there is $\delta > 0$ such that one of the following appears*

1. f is concave on $(x - \delta, x)$ and convex on $(x, x + \delta)$

2. f is convex on $(x - \delta, x)$ and concave on $(x, x + \delta)$.

Roughly speaking, the point x is a point of inflection if f changes from convex to concave or vice versa at point x.

Now we are ready to describe the problem of the course of function. The task 'examine the course of the following function' consists of the following sub-tasks:

1. To find out the domain, to determine whether the function is even, odd or periodic.
2. To find intersections with axes.
3. To examine the behavior of the function at the edges of the domain.
4. To derive function, to determine sets where the function is increasing and decreasing, to determine extremes.
5. To differentiate the function for the second time, to determine sets where the function is concave, convex, to determine points of inflection.
6. To sketch a graph of the function.

Let me comment each of this sub-tasks and let me use a function $f(x) = \frac{x^2+3}{x-1}$ as an example:

1. To determinate the domain one has to be sure that there is no division by 0, that the square root is taken from the non-negative number and that the argument of logarithm is positive. In case of the exemplary function we have to exclude the possibility of $x - 1 = 0$ which means that Dom $f = (-\infty, 1) \cup (1, \infty)$. Directly from the domain one may deduce that this function cannot be even, odd or periodic.
2. The intersections with axis are point of form $\langle 0, f(0)\rangle$ and $\langle x, 0\rangle$ where x solves $f(x) = 0$. In our case we obtain $\langle 0, -3\rangle$ and since

$$0 = \frac{x^2 + 3}{x - 1}$$

has no solution there is no intersection with axis x.

3. We have to evaluate limits on the edges of the domain. Let turn attention to our example. Since the domain is of the form $(-\infty, 1) \cup (1, \infty)$ we have to compute the following four limits:

$$\lim_{x \to -\infty} \frac{x^2+3}{x-1} = -\infty$$
$$\lim_{x \to 1-} \frac{x^2+3}{x-1} = -\infty$$
$$\lim_{x \to 1+} \frac{x^2+3}{x-1} = \infty$$
$$\lim_{x \to \infty} \frac{x^2+3}{x-1} = \infty$$

   Besides, we have to examine asymptotes.

   Definition 4.10. *Let* $\lim_{x\to\infty} \frac{f(x)}{x} = k_+ \in \mathbb{R}$ *and let* $\lim_{x\to\infty} f(x) - k_+x = q_+$. *Then an* a*symptote at* ∞ *is a line with equation* $y = k_+x + q_+$.

   Let $\lim_{x\to-\infty} \frac{f(x)}{x} = k_- \in \mathbb{R}$ *and let* $\lim_{x\to-\infty} f(x) - k_-x = q_-$. *Then an* a*symptote at* $-\infty$ *is a line with equation* $y = k_-x + q_-$.

   I our particular case we have:

$$\lim_{x \to \infty} \frac{x^2+3}{x-1}\frac{1}{x} = 1$$
$$\lim_{x \to \infty} \frac{x^2+3}{x-1} - x = 1$$
$$\lim_{x \to -\infty} \frac{x^2+3}{x-1}\frac{1}{x} = 1$$
$$\lim_{x \to -\infty} \frac{x^2+3}{x-1} - x = 1.$$

   So there is only line which represents asymptote at ∞ as well as at $-\infty$ and the equation of that line is

$$y = x + 1.$$

4. We have to differentiate the function and then we have to find all x such that $f'(x) > 0$ and all x for which $f'(x) < 0$. The points where the monotonicity of the function changes are extremal points.

   Take our exemplary function. We have $f'(x) = \frac{x^2-2x-3}{(x-1)^2}$. Consequently, $f'(x) > 0$ whenever $x \in (-\infty, -1)$ and $x \in (3, \infty)$. Moreover, $f'(x) < 0$ for $x \in (-1, 3) \setminus \{1\}$. Thus, f is increasing on $(-\infty, -1)$, f is decreasing on $(-1, 1)$, once again it is decreasing on $(1, 3)$ and f is increasing on $(3, \infty)$. We deduce that the local maximum is at point $x = -1$, its value is -2, the local minimum is at point $x = 3$, its value is 6.

5. We do the same as in the previous step but for the second derivative.

   Consider our exemplary function. We have $f''(x) = \frac{-4}{(x-1)^3}$. Consequently, $f''(x) < 0$ for $x \in (-\infty, 1)$ and $f''(x) > 0$ for $x \in (1, \infty)$ and f is concave on $(-\infty, 1)$ and convex on $(1, \infty)$. If 1 was a point of continuity of f, it would be a point of inflection. However, 1 does not belong to Dom f and thus there is no point of inflection.

6. Now we are ready to draw a graph using all the information we deduced.

4.3.3 Further use

The derivatives may be further used for computation of approximate values and for computation of limits. Without a proof, we state here two important concepts (this time we do not provide any proof):

Lemma 4.10 (l'Hospital). *Let f and g have finite derivatives for all $x \in (a,b) \subset \mathbb{R}$. Assume $g'(x) \neq 0$ and*

$$\lim_{x\to a+} \frac{f'(x)}{g'(x)} = A \in \mathbb{R}^*.$$

Let moreover one of the following is true:

1. $\lim_{x\to a+} f(x) = 0$ *and* $\lim_{x\to a+} g(x) = 0$ *or*
2. $\lim_{x\to a+} |g(x)| = \infty$.

Then

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = A.$$

Definition 4.11 (Taylor's sum). *Let f be $n-$times differentiable at point x_0. Then a polynomial of the form*

$$T_{f,x_0,n}(x) := f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}}{n!} = \sum_{j=0}^{n} \frac{f^{(i)}}{i!}(x-x_0)^i$$

is called the Taylor polynomial *to f at point x_0 of degree n.*

Lemma 4.11. *Assume that f is $(n+1)-$times differentiable at x_0. Let $x \in \mathbb{R}$ be arbitrary and let f is $(n+1)-$times differentiable on a closed interval I with edges at x_0 and x. Then there is ζ in between of x and x_0 such that*

$$f(x) - T_{f,x_0,n}(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-x_0)^{n+1}.$$

4.4 Integrals

Definition 4.12. *We say that F is an* antiderivative *(on interval (a,b)) of f if $F'(x) = f(x)$ (for all $x \in (a,b)$).*

We will use also the following notation

$$\int f(x)\, \mathrm{d}x = F(x).$$

Observation 4.7. *Let F_1 and F_2 be two antiderivatives of f on interval $(a,b) \subset \mathbb{R}$. Then $F_1 - F_2 \equiv c$ for some constant $c \in \mathbb{R}$.*

Proof. It suffices to consider $(F_1 - F_2)' = (f - f) = 0$. The claim follows immediately. □

As a consequence the antiderivative is not determined uniquely. In particular, the antiderivative to a given function f is a whole set of functions which differ by arbitrary constant – if F is an antiderivative of f then all functions in form $F + c$, $c \in \mathbb{R}$ are also antiderivatives.

4.4.1 Calculation – basic methods

Observation 4.8. *Let F be an antiderivative of f and G be an antiderivative of g. Then $F+G$ is an antiderivative of $f+g$ and cF is an antiderivative of cf for every $c \in \mathbb{R}$.*

Proof. It is enough to use rules for derivatives. $\square$

Further, we may use the table of basic derivatives in an 'inverted' way:

$f(x)$	$F(x)$	conditions
x^n	$\frac{1}{n+1}x^{n+1}+c,\ c\in\mathbb{R}$	$n \neq -1$, x as usual
x^{-1}	$\log\lvert x\rvert + c,\ c\in\mathbb{R}$	$x \neq 0$
e^x	$e^x + c,\ c\in\mathbb{R}$	$x\in\mathbb{R}$
a^x	$\frac{1}{\log a}a^x + c,\ c\in\mathbb{R}$	$x\in\mathbb{R}$, $a\in(0,1)\cup(1,\infty)$
$\sin x$	$-\cos x + c,\ c\in\mathbb{R}$	$x\in\mathbb{R}$
$\cos x$	$\sin x + c,\ c\in\mathbb{R}$	$x\in\mathbb{R}$
$\frac{1}{1+x^2}$	$\arctan x + c,\ c\in\mathbb{R}$	$x\in\mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c,\ c\in\mathbb{R}$	$x\in(-1,1)$

We present several exemplary calculations:

$$\int \frac{x+1}{\sqrt{x}}\ \mathrm{d}x = \int x^{\frac{1}{2}} + x^{-\frac{1}{2}}\ \mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c,\ \ c\in\mathbb{R},$$

$$\int \frac{x^2}{x^2+1}\ \mathrm{d}x = \int 1 - \frac{1}{1+x^2}\ \mathrm{d}x = x - \arctan x + c,\ \ c\in\mathbb{R},$$

$$\int \frac{2^{x+1}-5^{x-1}}{10^x}\ \mathrm{d}x = \int 2\left(\frac{1}{5}\right)^x + \frac{1}{5}\left(\frac{1}{2}\right)^x\ \mathrm{d}x = \frac{2}{\log\frac{1}{5}}\left(\frac{1}{5}\right)^x + \frac{1}{5\log\frac{1}{2}}\left(\frac{1}{2}\right)^x + c,\ \ c\in\mathbb{R}.$$

The first example of somewhat more advanced methods is 'linear substitution':

Observation 4.9. *Let $F(x)$ be an antiderivative to $f(x)$. Then $\frac{1}{a}F(ax+b)$ is an antiderivative of $f(ax+b)$.*

Proof. Indeed, we derive the composed function $F(ax+b)$:

$$(F(ax+b))' = F'(ax+b)(ax+b)' = f(ax+b)a.$$

$\square$

Below, we compute several exemplary exercises

$$\int (2x+3)^7\ \mathrm{d}x = \frac{1}{16}(2x+7)^8 + c,\ \ c\in\mathbb{R},$$

$$\int \frac{1}{x^2+4}\ \mathrm{d}x = \int \frac{1}{4}\frac{1}{(x/2)^2+1}\ \mathrm{d}x = \frac{1}{2}\arctan(x/2) + c,\ \ c\in\mathbb{R}.$$

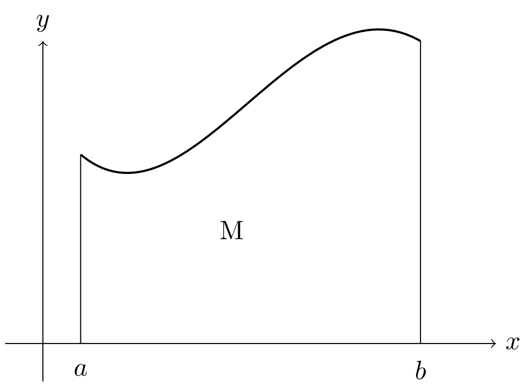
4.4.2 Riemann's integral

The main aim of this section is to compute the area which is bounded by a graph of function. More precisely, let f be a positive function defined on an interval (a, b). We will try to compute the area of a set

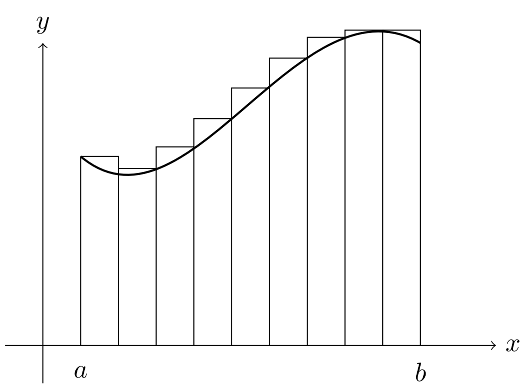
$$M = \{\langle x, y\rangle \in \mathbb{R}^2,\ x \in (a, b),\ 0 < y < f(x)\}. \tag{5}$$

The area is easy assuming $f \equiv c$, $c > 0$. In that case the area is given by $c(b - a)$.

In what follow, we show how to compute an area of the following set:

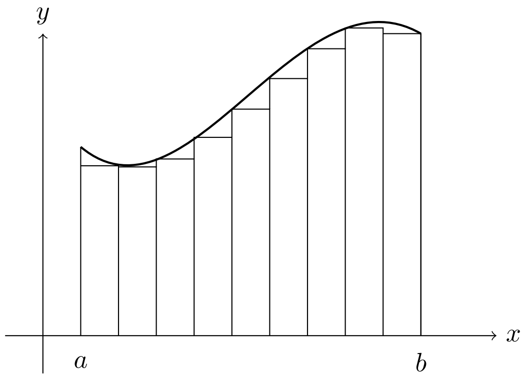


What if f is non-constant? We can approximate the value of the area by several rectangles as you can see on the following picture



Clearly, the area of M is less than the constructed approximation, however once there will be enough small rectangles, the approximation will be close to the true value.

We can also try to use the following approximation – this time we use maximal rectangle which are inside of the set M



In this case we obtain an area which is less than the area of M.
This idea is summarized in the following definition.

Definition 4.13. *Let f be a real function defined on $[a,b]$. We define sequences*

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{b-a}{n} \min\{f(x), x \in [a+(i-1)(b-a)/n, a+i(b-a)/n]\} \\ S_n &= \sum_{i=1}^{n} \frac{b-a}{n} \max\{f(x), x \in [a+(i-1)(b-a)/n, a+i(b-a)/n]\} \end{aligned} \tag{6}$$

If $\lim s_n = \lim S_n =: s$ then we say that s is the Riemann *integral of f over (a,b). We write*

$$(\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x = s$$

Let compute $\int_1^2 x^2\ \mathrm{d}x$: First, we divide $[1,2]$ to n subintervals with length $\frac{1}{n}$. Namely, the $i-$th subinterval is of the form

$$\left[1+\frac{i-1}{n}, 1+\frac{i}{n}\right].$$

Clearly, the maximum value of x^2 on this interval is $(1+\frac{i}{n})^2$, the minimum value is $(1+\frac{i-1}{n})^2$. We get

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{n}(1+\frac{i-1}{n})^2 \\ S_n &= \sum_{i=1}^{n} \frac{1}{n}(1+\frac{i}{n})^2 \end{aligned}$$

and we use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to obtain

$$\begin{aligned} s_n =& 1+\frac{n-1}{n}+\frac{(n-1)(2n-1)}{6n^2} \\ S_n =& 1+\frac{n+1}{n}+\frac{(n+1)(2n+1)}{6n^2} \end{aligned}$$

Since $\lim s_n = \lim S_n = \frac{7}{3}$ we deduce that this is the demanded area of the given set.

4.4.3 Newton's integral

Definition 4.14. *Let F be an antiderivative of f. Then*

$$(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x = F(b) - F(a).$$

The number $\int_a^b f(x)\ \mathrm{d}x$ is called the Newton integral of f over (a, b).

We use the notation $[F(x)]_a^b$ for the difference $F(b) - F(a)$.
The following theorem is presented without proof.

Theorem 4.1 (The basic theorem of calculus)**.** *Let f be defined on $[a, b]$ and let $(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x$ and $(\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x$ exist. Then*

$$(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x = (\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x$$

Let us note several remarks:

- This provides a simple way how to compute an area of the set M defined in (5).
- As the Riemann and Newton integrals are equal we write simply $\int_a^b f(x)\ \mathrm{d}x$ instead of $(\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x$ or $(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x$.

Definition 4.15. *Let f defined on (a, b) have antiderivative F. The number*

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{x\to a-} F(x) - \lim_{x\to b+} F(x)$$

*is called the*generalized *Newton integral of f over (a, b).*

Once again, we will use $[F(x)]_a^b$ for $\lim_{x\to a-} F(x) - \lim_{x\to b+} F(x)$.
With this at hand it is easy to compute areas of certain sets. Consider the integral from the previous section:

$$\int_1^2 x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}.$$

4.4.4 Calculation – method of substitution

We start with the method of substitution. The easiest example of substitution – a linear substitution – was presented in one of the previous lessons. Let me remind that if F is an antiderivative of f, then $\frac{1}{a}F(ax + b)$ is an antiderivative of $f(ax + b)$. This might be seen as a kind of substitution. In particular, we use substitution $y = ax + b$ (i.e., y is a linear function of x) and then $F'(y) = f(y)y' = f(y)a$.
Now we present a method how to use also a more general kind of substitution.
The formal description is the following:

Theorem 4.2. *Let $\varphi : (\alpha, \beta) \mapsto (a, b)$ has a finite derivative in every $x \in (\alpha, \beta)$ and let f be defined on (a, b). Then if F is an antiderivative of f then $F(\varphi)$ is an antiderivative of $f(\varphi) \cdot \varphi'$ on an (α, β).*

Proof. It follows from the rule of derivation of the composed function. We have

$$(F(\varphi))' = F'(\varphi)\varphi' = f(\varphi)\varphi'$$

□

Informally, we proceed as follows. Consider the following integral:

$$\int f(\varphi(x))\varphi'(x)\ \mathrm{d}x$$

We write

$$\begin{aligned}\varphi(x) &= t\\ \varphi'(x)\ \mathrm{d}x &= \mathrm{d}t\end{aligned}$$

and we plug everything to the given integral. Thus we get

$$\int f(\varphi(x))\varphi'(x)\ \mathrm{d}x = \int f(t)\ \mathrm{d}t = F(t) + c = F(\varphi(x)) + c.$$

We present several examples.

- $\int \sin^2 x \cos x\ \mathrm{d}x$. Here we use a substitution $t = \sin x$:

  $$\begin{aligned}\sin x &= t\\ \cos x\ \mathrm{d}x &= \mathrm{d}t.\end{aligned}$$

  We get

  $$\int \sin^2 x \cos x\ \mathrm{d}x = \int t^2\ \mathrm{d}t = \frac{t^3}{3} + c = \frac{\sin^3 x}{3} + c.$$

- $\int \frac{1}{x(\log^2 x+1)}\ \mathrm{d}x$. We use a substitution $t = \log x$:

  $$\begin{aligned}\log x &= t\\ \frac{1}{x}\ \mathrm{d}x &= \mathrm{d}t.\end{aligned}$$

  Thus,

  $$\int \frac{1}{x(\log^2 x + 1)}\ \mathrm{d}x = \int \frac{1}{t^2+1}\ \mathrm{d}t = \arctan t + c = \arctan(\log x) + c$$

- $\int x^3\sqrt{x^2+1}\ \mathrm{d}x$. The correct substitution is $t = x^2 + 1$. Note that then $x^2 = t - 1$:

  $$\begin{aligned}x^2 + 1 &= t\\ 2x\ \mathrm{d}x &= \mathrm{d}t.\end{aligned}$$

$$\begin{aligned}\int x^3\sqrt{x^2+1}\ \mathrm{d}x &= \frac{1}{2}\int x^2\sqrt{x^2+1}2x\ \mathrm{d}x = \frac{1}{2}\int (t-1)\sqrt{t}\ \mathrm{d}t\\ &= \frac{1}{2}\left(\frac{2}{5}t^{5/2} - \frac{2}{3}t^{3/2} + c\right) = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + c\end{aligned}$$

We may also use the substitution method to deduce the following observation which will be intensively used in the next subsection

Observation 4.10. *Any differentiable function f fulfills*

$$\int \frac{f'(x)}{f(x)}\ \mathrm{d}x = \log(|f(x)|) + c.$$

Proof. Indeed, it is enough to take $t = f(x)$. Then $f'(x)\ \mathrm{d}x = \mathrm{d}t$ and we have

$$\int \frac{f'(x)}{f(x)}\ \mathrm{d}x = \int \frac{1}{t}\ \mathrm{d}t = \log|t| + c = \log(|f(x)|) + c.$$

□

With this observation at hand it is easy to deduce, for example, that

$$\int \frac{2x+1}{x^2+x+3}\ \mathrm{d}x = \log|x^2+x+3| + c.$$

4.4.5 Calculation – integration by parts

The formal description of the integration by parts is mentioned in the following theorem.

Theorem 4.3. *Let F be an antiderivative of f and G be an antiderivative of g. Then*

$$\int f(x)G(x)\ \mathrm{d}x = F(x)G(x) - \int F(x)g(x)\ \mathrm{d}x.$$

Proof. It is enough to compute $\left(F(x)G(x) - \int F(x)g(x)\ \mathrm{d}x\right)'$:

$$\left(F(x)G(x) - \int F(x)g(x)\ \mathrm{d}x\right)' = f(x)G(x) + F(x)g(x) - F(x)g(x) = f(x)G(x).$$

This concludes the proof as we deduce that the derivative of the right hand side is equal to $f(x)G(x)$. □

Clearly, this technique is useful (not only) for integration of a product of two functions. The most typical situation is polynomial × sin, cos or exponential function. Below we present some typical cases.

- Polynomial times exp, sin, cos. In this case it is enough to lower the degree of the polynomial to 0 by several use of the integration by parts. Let compute

  $$\int x^2 e^{3x}\ \mathrm{d}x.$$

  We take $G(x) = x^2$ and $f(x) = e^{3x}$. This gives $g(x) = 2x$ and $F(x) = \frac{1}{3}e^{3x}$. We get

  $$\int x^2 e^{3x}\ \mathrm{d}x = x^2\frac{1}{3}e^{3x} - \frac{1}{3}\int 2xe^{3x}\ \mathrm{d}x$$

  and we use integration by parts once again with $G(x) = 2x$ and $f(x) = e^{3x}$ to get

  $$\int x^2 e^{3x}\ \mathrm{d}x = x^2\frac{1}{3}e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{9}\int e^{3x}\ \mathrm{d}x$$

  which gives

  $$\int x^2 e^{3x}\ \mathrm{d}x = x^2\frac{1}{3}e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + c,\ c \in \mathbb{R}.$$

- sin, cos, exp times sin, cos, exp. This time we have to integrate by parts twice in a row and then we have to use the resulting equation to compute the original integral. This is illustrated in the exercise below:
  Compute
  $$\int e^{2x} \sin x \; \mathrm{d}x$$
  We take $G(x) = e^{2x}$ and $f(x) = \sin x$. This gives $g(x) = 2e^{2x}$ and $F(x) = -\cos x$. Thus
  $$\int e^{2x} \sin x \; \mathrm{d}x = -e^{2x} \cos x + \int 2e^{2x} \cos x \; \mathrm{d}x.$$
  We apply integration by parts once again with $G(x) = e^{2x}$ and $f(x) = \cos x$ (yielding $g(x) = 2e^{2x}$ and $F(x) = \sin x$) to get
  $$\int e^{2x} \sin x \; \mathrm{d}x = -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \; \mathrm{d}x.$$
  This gives
  $$\int e^{2x} \sin x \; \mathrm{d}x = -\frac{1}{5} e^{2x} \cos x + \frac{2}{5} e^{2x} \sin x + c, \; c \in \mathbb{R}.$$
- functions with rational derivatives. Here we have to always differentiate a function which has a nice rational derivative:
  Compute
  $$\int \arctan x \; \mathrm{d}x$$
  We may write $\arctan x$ as $1 \cdot \arctan x$ and take $G(x) = \arctan x$ and $f(x) = 1$. This choice gives $g(x) = \frac{1}{1+x^2}$ and $F(x) = x$.
  $$\int \arctan x \; \mathrm{d}x = x \arctan x - \int \frac{x}{x^2+1} \; \mathrm{d}x.$$
  Now we use substitution $x^2 + 1 = t$, $2x \; \mathrm{d}x = \mathrm{d}t$ and thus
  $$\int \frac{x}{x^2+1} \; \mathrm{d}x = \int \frac{1}{2} \frac{1}{t} \; \mathrm{d}t = \frac{1}{2} \log |t| + c = \frac{1}{2} \log(x^2+1) + c$$
  To sum up we get
  $$\int \arctan x \; \mathrm{d}x = x \arctan x - \frac{1}{2} \log(x^2+1) + c.$$
  As next example, compute
  $$\int x \log x \; \mathrm{d}x.$$
  Once again, take $G(x) = \log x$ and $f(x) = x$. This gives $g(x) = \frac{1}{x}$ and $F(x) = \frac{1}{2}x^2$. We get
  $$\int x \log x \; \mathrm{d}x = \frac{1}{2} x^2 \log x - \frac{1}{2} \int x^2 \frac{1}{x} \; \mathrm{d}x = \frac{1}{2} x^2 \log x - \frac{1}{4} x^2 + c, \; c \in \mathbb{R}.$$

4.4.6 Integration of rational functions

This section deals with integrals of the form

$$\int \frac{P(x)}{Q(x)}\, \mathrm{d}x$$

where P and Q are polynomials.
Moreover, we may assume $\deg P < \deg Q$. In case this is not true it is enough to divide P by Q:

For example let compute $\int \frac{x^3+4x}{x^2+2}\, \mathrm{d}x$. It holds that

$$\frac{x^3+4x}{x^2+2} = x + \frac{2x}{x^2+2}$$

and thus

$$\int \frac{x^3+4x}{x^2+2}\, \mathrm{d}x = \int x + \frac{2x}{x^2+2}\, \mathrm{d}x = \frac{x^2}{2} + \log(x^2+2) + c,\ c \in \mathbb{R}$$

Below we present the partial fraction decomposition. It starts with the following theorem.

Theorem 4.4. *Every polynomial can be written as a product of* $1-$*degree polynomials and irreducible* $2-$*degree polynomials.*

Recall that a polynomial $ax^2 + bx + c$ is irreducible if there are no real roots.
We adopt the following strategy: the polynomial Q in the denominator may be written as a product of the aforementioned polynomials. In that case, the whole fraction is rewritten as a sum of fractions with $1-$ and $2-$ degree polynomials in the denominator (partial fraction decomposition). This sum may be integrated by methods mentioned in the previous talks.

Below we show how to deal with $1-$degree polynomials. Let compute $\int \frac{x+1}{x^2+5x+6}\, \mathrm{d}x$. We know that $(x^2+5x+6) = (x+2)(x+3)$ and thus

$$\begin{aligned}\frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)} &= \frac{A}{x+2} + \frac{B}{x+3} \\ &= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}.\end{aligned}$$

This yields

$$x + 1 = Ax + 3A + Bx + 2B$$

and we compare appropriate coefficients to deduce

$$\begin{aligned}1 &= A + B \\ 1 &= 3A + 2B\end{aligned}$$

Thus $A = -1$ and $B = 2$ and, consequently,

$$\frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)} = \frac{2}{x+3} - \frac{1}{x+2}.$$

Thus

$$\int \frac{x+1}{x^2+5x+6}\, \mathrm{d}x = 2\int \frac{1}{x+3}\, \mathrm{d}x - \int \frac{1}{x+2}\, \mathrm{d}x = 2\log|x+3| - \log|x+2| + c,\ c \in \mathbb{R}$$

What happens if there is a one-degree polynomial powered to some number higher than 1? This is shown in the following example. Let compute $\int \frac{3x^2-2x}{(x-1)^2(2x-1)}\,\mathrm{d}x$. This time we write

$$\frac{3x^2-2x}{(x-1)^2(2x-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x-1}$$
$$= \frac{A(x-1)(2x-1) + B(2x-1) + C(x-1)^2}{(x-1)^2(2x-1)}.$$

We deduce

$$3x^2 - 2x = A(2x^2 - 3x + 1) + B(2x-1) + C(x^2 - 2x + 1)$$

and thus

$$\begin{aligned} 3 &= 2A + C \\ -2 &= -3A + 2B - 2C \\ 0 &= A - B + C \end{aligned}$$

This yields $A = 2$, $B = 1$ and $C = -1$ and

$$\int \frac{3x^2-2x}{(x-1)^2(2x-1)}\,\mathrm{d}x = 2\int \frac{1}{x-1}\,\mathrm{d}x + \int \frac{1}{(x-1)^2}\,\mathrm{d}x - \int \frac{1}{2x-1}\,\mathrm{d}x$$
$$= 2\log|x-1| - (x-1)^{-1} - \frac{1}{2}\log|2x-1| + c,\ c \in \mathbb{R}.$$

We turn our attention to an irreducible polynomial of degree 2 First we show how to integrate such functions:

$$\int \frac{2x+3}{x^2+4x+8}\,\mathrm{d}x = \int \frac{2x+4}{x^2+4x+8}\,\mathrm{d}x - \int \frac{1}{x^2+4x+8}\,\mathrm{d}x.$$

The first integral on the right hand side is of the form $\frac{f'}{f}$ and we use Observation 4.10 to get

$$\int \frac{2x+4}{x^2+4x+8}\,\mathrm{d}x = \log|x^2+4x+8| + c,\ c \in \mathbb{R}.$$

Further, we have

$$\int \frac{1}{x^2+4x+8}\,\mathrm{d}x = \int \frac{1}{x^2+4x+4+4}\,\mathrm{d}x = \int \frac{1}{(x+2)^2+4}\,\mathrm{d}x.$$

This can be rearranged to some kind of arctan:

$$\int \frac{1}{(x+2)^2+4}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}+1\right)^2+1}\,\mathrm{d}x = \frac{1}{2}\arctan\left(\frac{x}{2}+1\right) + c,\ c \in \mathbb{R}$$

and we obtain

$$\int \frac{2x+3}{x^2+4x+8}\,\mathrm{d}x = \log|x^2+4x+8| - \frac{1}{2}\arctan\left(\frac{x}{2}+1\right) + c,\ c \in \mathbb{R}.$$

Now we can perform the partial fraction decomposition. Let compute $\int \frac{6x+4}{(x^2+2x+2)(x-1)}\ \mathrm{d}x$. This time we have

$$\frac{6x+4}{(x^2+2x+2)(x-1)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+C(x^2+2x+2)}{(x^2+2x+2)(x-1)}$$

and we deduce

$$\begin{aligned} 0 &= A+C \\ 6 &= -A+B+2C \\ 4 &= -B+2C \end{aligned}$$

We deduce $A=-2$, $B=0$ and $C=2$ and thus

$$\int \frac{6x+4}{(x^2+2x+2)(x-1)}\ \mathrm{d}x = 2\int \frac{1}{x-1}\ \mathrm{d}x - \int \frac{2x}{x^2+2x+2}\ \mathrm{d}x$$

The first integral is simple so we pay attention to the second one. We have

$$\begin{aligned} \int \frac{2x}{x^2+2x+2}\ \mathrm{d}x = \int \frac{2x+2}{x^2+2x+2}\ \mathrm{d}x - 2\int \frac{1}{(x+1)^2+1}\ \mathrm{d}x \\ = \log(x^2+2x+2) - 2\arctan(x+1) + c,\ c\in\mathbb{R} \end{aligned}$$

Thus

$$\int \frac{6x+4}{(x^2+2x+2)(x-1)}\ \mathrm{d}x = 2\log|x-1| - \log(x^2+2x+2) + 2\arctan(x+1) + c,\ c\in\mathbb{R}.$$

The above attitude (partial fraction decomposition) may be summarized as follows

Theorem 4.5. *Let* $\deg P < \deg Q$ *and let*

$$Q(x) = \alpha_0(x-\alpha_1)^{r_1}\cdot\ldots\cdot(x-\alpha_k)^{r_k}(x^2+p_1x+q_1)^{s_1}\cdot\ldots\cdot(x^2+p_lx+q_l)^{s_l}$$

where the second order polynomials have no real roots and no multiplier divide any other one and all coefficients are integers. Then there are real numbers $A_{11},\ldots,A_{1r_1},\ldots,A_{k1},\ldots,A_{kr_k}$ *and* $B_{11},C_{11},\ldots,B_{1s_1},C_{1s_1},\ldots B_{l1},C_{l1},\ldots,B_{ls_l},C_{ls_l}$ *such that*

$$\begin{aligned} \frac{P(x)}{Q(x)} = \frac{A_{11}}{x-\alpha_1} + \ldots + \frac{A_{1r_1}}{(x-\alpha_1)^{r_1}} + \ldots + \frac{A_{k1}}{(x-\alpha_k)} \\ + \ldots + \frac{A_{kr_k}}{(x-\alpha_k)^{r_k}} + \frac{B_{11}x+C_{11}}{x^2+p_1x+q_1} + \ldots + \frac{B_{1s_1}x+C_{1s_1}}{(x^2+p_1x+q)^{s_1}} \\ + \ldots + \frac{B_{l1}x+C_{l1}}{x^2+p_lx+q} + \ldots + \frac{B_{ls_l}x+C_{ls_l}}{(x^2+p_lx+q_l)^{s_l}} \end{aligned}$$

This theorem is presented without a proof.

The above procedure may be applied also on integrals containing sin and cos. In particular, we consider integrals of form

$$\int R(\sin(x),\cos(x))\ \mathrm{d}x$$

where R is a rational function to integrals which might be solved out by the partial fraction decomposition.

In case everything appears in an even power, you can use substitution $\tan x = t$. In that case $\mathrm{d}x = \frac{1}{1+t^2}\ \mathrm{d}t$, $\cos^2 x = \frac{1}{t^2+1}$ and $\sin^2 x = \frac{t^2}{t^2+1}$. Thus

$$\begin{aligned}\int \frac{sin^2 x}{1+\sin^2 x}\ \mathrm{d}x &= \int \frac{t^2}{2t^2+1}\frac{1}{t^2+1}\ \mathrm{d}t \\ &= \int \frac{1}{t^2+1}\ \mathrm{d}t - \int \frac{1}{2t^2+1}\ \mathrm{d}t = \arctan t - \frac{\sqrt{2}}{2}\arctan(\sqrt{2}t) + c \\ &= x - \frac{\sqrt{2}}{2}\arctan(\sqrt{2}\tan x) + c,\ c \in \mathbb{R}\end{aligned}$$

If there are also other powers of sin or cos, we use a universal substitution $\tan\frac{x}{2} = t$. In that case $\mathrm{d}x = \frac{2}{1+t^2}\ \mathrm{d}t$, $\sin x = \frac{2t}{t^2+1}$ and $\cos x = \frac{1-t^2}{t^2+1}$. Compute, for example,

$$\begin{aligned}\int \frac{1}{2\sin x - \cos x + 5}\ \mathrm{d}x &= \int \frac{1}{\frac{4t}{t^2+1} - \frac{1-t^2}{t^2+1} + 5}\frac{2}{1+t^2}\ \mathrm{d}t \\ &= \int \frac{1}{4t - 1 + t^2 + 5t^2 + 5}\ \mathrm{d}t = \int \frac{1}{6t^2+4t+4}\ \mathrm{d}t \\ &= \frac{1}{\sqrt{20}}\arctan\left(\frac{3}{\sqrt{5}}\tan\frac{x}{2} + \frac{1}{\sqrt{5}}\right) + c,\ c \in \mathbb{R}.\end{aligned}$$

4.4.7 Use of integrals

Length of curve:

$$L(c) = \int_a^b \sqrt{1 + (f'(x))^2}\ \mathrm{d}x$$

where c is a graph of function f for $x \in (a, b)$.

Compute the length of a curve $f(x) = x^{\frac{3}{2}}$, $x \in [0, 4]$. We have $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$ and thus

$$\begin{aligned}\int_0^4 \sqrt{1+\frac{9}{4}x}\ \mathrm{d}x = \int_0^4 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}}\ \mathrm{d}x = \frac{8}{27}\left[\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^4 \\ = \frac{8}{27}\left(\sqrt{10}^3 - 1\right)\end{aligned}$$

Volume of a solid of revolution:

Let take a set

$$M = \{\langle x, y\rangle \in \mathbb{R},\ x \in (a, b),\ 0 \le y < f(x)\}$$

and rotate it around the axis x. The volume of the solid that has arisen is

$$V = \pi \int_a^b f^2(x)\ \mathrm{d}x$$

Let compute the volume of cone which arises as a rotation of the set $M = \{0 \le y \le x, x \in (0, 4)\}$ around the axis x:

$$\pi \int_0^4 x^2\ \mathrm{d}x = \pi\left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3}\pi.$$

4.5 Improper integrals

Recall, that the generalized Newton integral is defined as

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{x\to b_-} F(x) - \lim_{x\to a_+} F(x)$$

where F is an antiderivative of f. This allows to compute integrals where a or b are $\pm\infty$ or where $F(a)$ are not defined.

Let compute $\int_0^\infty \frac{1}{1+x^2}\ \mathrm{d}x$. We have

$$\int_0^\infty \frac{1}{1+x^2}\ \mathrm{d}x = [\arctan x]_0^\infty = \lim_{x\to\infty} \arctan x - \lim_{x\to 0_-} \arctan x = \frac{\pi}{2}$$

Let compute $\int_0^1 \frac{1}{\sqrt{x}}\ \mathrm{d}x$. We have

$$\int_0^1 x^{-1/2}\ \mathrm{d}x = \left[2x^{1/2}\right]_0^1 = 2 - 0 = 2.$$

But

$$\int_0^1 \frac{1}{x^2}\ \mathrm{d}x = \left[-x^{-1}\right]_0^1 = 1 + \infty = \infty$$

and

$$\int_1^\infty x\ \mathrm{d}x = \left[\frac{1}{2}x^2\right]_1^\infty = \infty$$

In what follows we consider a problem of 'finiteness' of improper integrals.

Definition 4.16. *Let $\int_a^b f(x)\ \mathrm{d}x$ is finite. Then we say, that this integral* converges. *In other case we say it* diverges.

Clearly, let $-\infty < a < b < \infty$ and $f(x)$ is bounded on (a, b). Then $\int_a^b f(x)\ \mathrm{d}x$ converges.

Comparison: let f and g be nonnegative function on an interval $(a, b) \subset \mathbb{R}$. Then:

- if $f \le g$ and $\int_a^b g(x)\ \mathrm{d}x$ converges then also $\int_a^b f(x)\ \mathrm{d}x$ converges.
- if $f \le g$ and $\int_a^b f(x)\ \mathrm{d}x$ diverges then also $\int_a^b g(x)\ \mathrm{d}x$ diverges.

Does $\int_3^\infty \frac{1}{x^2+2x}\ \mathrm{d}x$ converge?

Yes, because $\frac{1}{x^2+2x} \le \frac{1}{x^2}$ for all $x \in (3, \infty)$ and

$$\int_3^\infty \frac{1}{x^2}\ \mathrm{d}x = \left[-\frac{1}{x}\right]_3^\infty = -\lim_{x\to\infty} \frac{1}{x} + \frac{1}{3} = \frac{1}{3}.$$

How about integral $\int_3^\infty \frac{1}{x^2-2x}\ \mathrm{d}x$? This time we have $\frac{1}{x^2-2x} \ge \frac{1}{x^2}$ but that is not sufficient. We postpone this question.

Scale: Consider an integral

$$\int_0^1 \frac{1}{x^n}\ \mathrm{d}x$$

This integral converges for $n < 1$ and diverges for $n \geq 1$.
Consider

$$\int_1^{\infty} \frac{1}{x^n}\ \mathrm{d}x$$

This integral converges for $n > 1$ and diverges for $n \leq 1$.

Comparison, limit case: Let $f,\ g$ be continuous on $[a,b) \subset \mathbb{R}$. Let

$$\lim_{x \to b_-} \frac{f(x)}{g(x)} \in (0, \infty).$$

Then $\int_a^b f(x)\ \mathrm{d}x$ converges if and only if $\int_a^b g(x)\ \mathrm{d}x$ converges.

Now we can decide about the convergence of

$$\int_3^{\infty} \frac{1}{x^2 - 2x}\ \mathrm{d}x.$$

4.6 Exercises

1. Define one-sided infinite limits.

2. Define infinite limits at infinity.

3. Use definition of limit to proof $\lim_{x \to \infty} \frac{1}{x} = 0$.

4. Try to find functions $f, g : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{x \to x_0} g(x) = A \in \mathbb{R}$, $\lim_{x \to A} f(x) = B \in \mathbb{R}$ and, simultaneously, $\lim_{x \to x_0} f(g(x)) \neq B$.

5. Deduce (4) from the basic definitions of elementary functions.

6. Try to find functions $f, g : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{x \to x_0} g(x) = A \in \mathbb{R}$, $\lim_{y \to A} f(y) = B \in \mathbb{R}$ and, simultaneously, $\lim_{x \to x_0} f(g(x)) \neq B$.

7. Compute the following limit of sequence

$$\lim \left(1 + \frac{1}{n}\right)^n.$$

8. If f is continuous at point x_0 then it has a real derivative at x_0. Is this claim true?

9. Compute $(x^x)'$ where $x \in (0, \infty)$.

10. Try to compute the length of half of the circle, i.e. the length of the graph of the function $y = \sqrt{1 - x^2}$ where $x \in \langle -1, 1 \rangle$. Hint: use $x = \sin t$.

11. Try to compute the volume of the torus, this means the volume of the solid which arises (for example) as a rotation of the set $M = \{x \in (-1,1),\ -\sqrt{1-x^2} < y - 3 < \sqrt{1+x^2}\}$ around the axis x.

5 Differential equations

5.1 Introduction

Recall, that for $y(x) : \mathbb{R} \mapsto \mathbb{R}$ the meaning of $y'(x)$ is 'increment'. This section is devoted to the study of *differential equations*, i.e., equations where the increment depends on y.

We start with a simple example. Assume that $y(x)$ denotes the number of people infected by some disease. Then we may deduce that the increment $y'(x)$ of this value depends linearly on the number of people infected. Thus,

$$y'(x) = ky(x)$$

for some constant $k \in (0, \infty)$ (usualy this constant has to be determined with the help of known data and in collaboration with other experts). We deduce that

$$\frac{y'(x)}{y(x)} = k$$

and, consequently,

$$(\log y(x))' = k$$

which gives $y(x) = ce^{kx}$ for some constant $c \in \mathbb{R}$.

Assume we have a differential equation of the form

$$y' = f(x, y) \tag{7}$$

Here of course $y = y(x)$, however we will omit this dependency to shorten the notation.

Definition 5.1. *A solution to this differential equation on an interval $(c, d) \subset \mathbb{R}$ is every function $y(x)$ which fulfills (7). The solution of (7) on (c, d) which might not be prolonged to some $(c', d') \supset (c, d)$ is called a* maximal solution. *The set of all maximal solutions is called a* general solution.

5.2 Separation of variables

This subsection is devoted to the study of equations of the following form:

$$y'(x) = f(y(x))g(x).$$

First of all, let $y_0 \in \mathbb{R}$ be such that $f(y_0) = 0$. Then

$$y \equiv y_0$$

is a *stationary solution*.

Assuming y is such that $f(y) \neq 0$, we can deduce

$$\frac{y'(x)}{f(y(x))} = g(x).$$

Assume $F(y)$ is antiderivative of $\frac{1}{f(y)}$. Then the equation might be rewritten as

$$F'(y) = g(x)$$

and we deduce that

$$F(y(x)) = G(x)$$

where $G' = g$.

It it is possible, we deduce how y depends on x.

Let solve

$$y' = 2\sqrt{|y|}.$$

Sticking: Let $y_1 : (a, x_0) \mapsto \mathbb{R}$ and $y_2 : (x_0, b) \mapsto \mathbb{R}$ are solutions to

$$y' = f(x, y).$$

Let $\lim_{x \to x_0-} y_1 = \lim_{x \to x_0+} y_2 = y_0$ and let $f(x, y)$ is continuous at (x_0, y_0). Then

$$y = \begin{cases} y_1 \text{ for } x \in (a, x_0) \\ y_0 \text{ for } x = x_0 \\ y_2 \text{ for } x \in (x_0, b) \end{cases}$$

is a solution to $y' = f(x, y)$ on (a, b).
This allows to conclude the example.

Further equations which can be solved by the separation of variables:
Equations of the form

$$y' = f(x, y)$$

where $f(\lambda x, \lambda y) = f(x, y)$ for every $\lambda \neq 0$. In that case we can write $f(x, y) = f(1, \frac{y}{x})$ and we use a substitution $z = \frac{y}{x}$. As $y = zx$, we get $y' = z'x + z$ and the equation might be rewritten as

$$z'x + z = f(1, z)$$

which yields (in case everything is well defined)

$$z' = (f(1, z) - z)\frac{1}{x}.$$

This equation can be solved by the separation of variables.

Example: Let treat $y' = \frac{-xy+x^2+y^2}{x^2}$. We use substitution $y(x) = xz(x)$ (i.e. $z(x) = \frac{y(x)}{x}$. We obtain

$$xz' + z = -z + 1 + z^2$$

yielding

$$xz' = (z - 1)^2.$$

5.3 First-order linear equations

This subsection deals with equations of the form

$$y'(x) + a(x)y(x) = b(x) \tag{8}$$

where y is an unknown function and a and b are given functions. Such equations are called linear because y as well as y' appears only in the first power. We denote the left hand side of (8) by $L(y)$, i.e.

$$L(y) = y'(x) + a(x)y(x). \tag{9}$$

L is an operator (the function whose input is function) and it is linear, meaning

$$
\begin{aligned}
L(y_1 + y_2) &= L(y_1) + L(y_2) \\
L(\alpha y_1) &= \alpha L(y_1)
\end{aligned}
\tag{10}
$$

for every $\alpha \in \mathbb{R}$ and every pair of differentiable functions y_1 and y_2. There are two ways how to obtain a solution to this equation. One is called 'variation of constants', the second one is called 'integration factor'.

5.3.1 Variation of constants

We start by a simple observation

Observation 5.1. *Let y_1 and y_2 be two solutions to* (8)*. Then $y_1 - y_2$ solves* (8) *with $b(x) \equiv 1$.*

Proof. Due to the linearity of L we have

$$L(y_1 - y_2) = L(y_1) - L(y_2) = b(x) - b(x) = 0.$$

□

The linear differential equation with zero right hand side (i.e., (8) with $b(x) \equiv 0$) is called homogeneous linear differential equation.

Observation 5.2. *Let $M = \{y, L(y) = 0\}$ be the set of all solutions to a homogeneous differential equation and let y_p be a particular solution to $L(y) = b(x)$. Then $\{y_p + y,\ y \in M\}$ is the set of all solutions to $L(y) = b(x)$.*

Proof. If y is such that $L(y) = 0$ and y_p satisfies $L(y_p) = b(x)$ then $L(y + y_p) = b(x)$.

Next let y_r be an arbitrary function satisfying $L(y_r) = b(x)$. Then $L(y_r - y_p) = L(y_r) - L(y_p) = b(x) - b(x) = 0$ and thus $y_r - y_p \in M$ and, consequently, y_r can be written as $y_r = y_p + (y_r - y_p)$. □

This observation provides a manual how to find all solutions to the given first order linear differential equation. First, we find all solutions to

$$y'(x) + a(x)y(x) = 0.$$

This can be done by the separation of variables. By this we obtain that

$$y(x) = cg(x) \tag{11}$$

for some function g.

It remains to find one particular solution to

$$y'(x) + a(x)y(x) = b(x).$$

In order to do so, we use the method called 'variation of parameters'. We assume that one solution is of the form $y(x) = c(x)g(x)$ (we just assume that the constant appearing in (11) is actually a function of x). We have $y'(x) = c'(x)g(x) + c(x)g'(x)$ and since $g(x)$ is a solution to the homogeneous problem, we deduce

$$c'(x)g(x) = b(x).$$

which is an equation for $c'(x)$. The solution is then $c(x) = \int \frac{b(x)}{g(x)}\ \mathrm{d}x$.

Example: Let solve

$$y' + y = \sin x e^{-x}. \tag{12}$$

The appropriate homogeneous equation is of the form

$$y' + y = 0$$

which gives $y = ce^{-x}$, $c \in \mathbb{R}$. Let now assume that one solution is of the form $y = c(x)e^{-x}$. We have $y'(x) = c'(x)e^{-x} - c(x)e^{-x}$. We plug this into (12) to obtain

$$c'(x)e^{-x} = \sin x e^{-x}.$$

This yields $c'(x) = \sin x$ and thus $c(x) = -\cos x$. As a result, the demanded particular solution is $y = -\cos x e^{-x}$ and all solutions to (12) are of the form

$$y = -\cos x e^{-x} + ce^{-x}, \quad c \in \mathbb{R}.$$

5.3.2 Integration factor

We multiply (8) by certain function $A(x)$ (integration factor) to obtain

$$A(x)y'(x) + A(x)a(x)y(x) = A(x)b(x).$$

If A is chosen correctly then the left hand side will become a derivation of a function z defined as $z = A(x)y(x)$. This is true as far as $A'(x) = A(x)a(x)$. Thus

$$z'(x) = A(x)b(x)$$

and the integration factor $A(x)$ is equal to $e^{\int a(x)\ \mathrm{d}x}$. (Recall it holds that $\left(e^{\int a(x)\ \mathrm{d}x}\right)' = a(x)e^{\int a(x)\ \mathrm{d}x} = a(x)A(x)$)

Exercise Let solve

$$xy' - 3y = x^4$$

First, we have to assume $x \neq 0$ (we will solve the equation separately on $(-\infty, 0)$ and $(0, \infty)$) in order to rearrange the equation to

$$y' - \frac{3}{x}y = x^3. \tag{13}$$

We have

$$\int -\frac{3}{x}\ \mathrm{d}x = -3 \log|x| = \log|x|^{-3}$$

and, consequently, the desired integrating factor is $e^{\log|x|^{-3}} = |x|^{-3}$. Let solve the equation on $(0, \infty)$. We multiply (13) by x^{-3} to get

$$x^{-3}y' - 3x^{-4}y = 1$$

and we use a substitution $z = x^{-3}y$. Then $z' = x^{-3}y' - 3x^{-4}y$ (this is effective also on $(-\infty, 0)$ and (13) becomes

$$z' = 1.$$

This yields

$$z = x + c,\ c \in \mathbb{R}$$

and since $y = zx^3$ we obtain

$$y = x^3(x + c),\ c \in \mathbb{R},\ x \in (0, \infty) \text{ or } x \in (-\infty, 0). \tag{14}$$

Sticking: In the last step we have to find all maximal solutions to the original equation, i.e.

$$xy' - 3y = x^4$$

note that this equation is well defined also for $x = 0$. We take solutions (**??**). Note that a solution y_1 defined on $(-\infty, 0)$ can be extended by a solution y_2 defined on $(0, \infty)$ if $\lim_{x\to 0_-} y_1(x) = \lim_{x\to 0_+} y_2(x)$ and $\lim_{x\to 0_-} y_1'(x) = \lim_{x\to 0_+} y_2'(x)$. Here we have $\lim_{x\to 0_-} y(x) = \lim_{x\to 0_+} y(x) = \lim_{x\to 0_-} y'(x) = \lim_{x\to 0_+} y'(x) = 0$. Thus, all solutions to this equation are of the form

$$y(x) = \begin{cases} x^3(x + c_1),\ x \in (-\infty, 0] \\ x^3(x + c_2),\ x \in (0, \infty) \end{cases}$$

where $c_1,\ c_2 \in \mathbb{R}$ are arbitrary constants.

5.4 Linear equations with constant coefficients

Let $n \in N$. The $n-$th order linear equation with constant coefficient is an equation of the form

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = b(x) \tag{15}$$

where $a_i \in \mathbb{R}$ for all $i \in \{0, 1, \ldots, n\}$, $a_n \neq 0$, $y(x)$ is an unknown $n-$th times differentiable function and $b(x)$ is given right hand side.
For example,

$$y''' + 4y'' - 5y = e^x$$

is a third-order linear equation.

We denote the left hand side of the equation by $L(y)$, i.e.,

$$L(y) = a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x)$$

The operator $L(y)$ is linear:

$$\begin{aligned} L(y_1 + y_2) &= L(y_1) + L(y_2) \\ L(a y_1) &= a L(y_1) \end{aligned}$$

for all $n-$th times differentiable functions y_1 and y_2 and for all $a \in \mathbb{R}$.

Similarly as in the previous subsection we may deduce

Observation 5.3. *Let y_1 and y_2 be two solutions to* (15)*. Then $y_1 - y_2$ solves* (15) *with zero right hand side.*

and

Observation 5.4. *Let $M = \{y, L(y) = 0\}$ be the set of solutions to a homogeneous differential equation and let y_p be a particular solution to $L(y) = b$. Then $\{y_p + y, y \in M\}$ is the set of solutions to $L(y) = b$.*

This is once again the way how to find all solutions to the non-homogeneous linear differential equation. First, we treat the appropriate homogeneous problem and we find all its solutions. Next, we find one solution to the non-homogeneous problem. The sum of this particular solution and all solutions to the homogeneous problem is then the set of all solutions to the non-homogeneous problem.

We treat the problem

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = 0 \tag{16}$$

We assume that the solution is in the form $y(x) = e^{\lambda x}$. We obtain the following equation for unknown λ

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0 \tag{17}$$

This is called characteristic equation.

Assume now that the characteristic equation has n different roots λ_1, λ_2, …, λ_n. Then the solution is every function of the form

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \ldots + c_n e^{\lambda_n x}$$

where $c_1,\ c_2,\ \ldots,\ c_n \in \mathbb{R}$ are arbitrary constants. The set

$$\{e^{\lambda_1 x},\ e^{\lambda_2 x}, \ldots,\ e^{\lambda_n x}\}$$

is called fundamental system.

Example Consider the equation

$$y'' + 4y' + 3y = 0.$$

Its characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0$$

and it has two solutions, $\lambda_1 = -1$ and $\lambda_2 = -3$. Consequently, the set $\{e^{-x}, e^{-3x}\}$ is the fundamental system and all solutions to the given equation are of the form $y(x) = c_1 e^{-x} + c_2 e^{-3x}$ where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

What happens if there are complex roots? If $\lambda = \mu + i\nu$ is a root then $\overline{\lambda} = \mu - i\nu$ is also a root. Functions in fundamental system which are related to this pair of roots are

$$e^{\mu x}\sin(\nu x),\ e^{\mu x}\cos(\nu x).$$

And what if λ is a root with multiplicity higher than 1 (say multiplicity is equal to l)? Then, in case λ is real, the fundamental system contains functions

$$e^{\lambda x}, xe^{\lambda x}, \ldots, x^{l-1}e^{\lambda x}.$$

If $\lambda = \mu + i\nu$ then the fundamental system contains

$$e^{\mu x}\sin(\nu x),\ e^{\mu x}\cos(\nu x), xe^{\mu x}\sin(\nu x),\ xe^{\mu x}\cos(\nu x),$$
$$\ldots, x^{l-1}e^{\mu x}\sin(\nu x),\ x^{l-1}e^{\mu x}\cos(\nu x).$$

Once we compute all solutions to the homogeneous we need to find at least one particular solution. This is simple once the right hand side of the equation is of special form. This is described in the following theorem:

Theorem 5.1. *Consider an equation*

$$L(y) = e^{\alpha x}[q_1(x)\cos(\beta x) + q_2(x)\sin(\beta x)] \tag{18}$$

where q_1 and q_2 are polynomials of degree at most $n \in \mathbb{N}$. Let $k \geq 0$ be the multiplicity of a root $\lambda = \alpha + \beta i$ of the characteristic equation (take $k = 0$ if such λ is not a root).

Then there are uniquely determined polynomials r_1 and r_2 of degree at most n such that

$$y = x^k e^{\alpha x}[r_1(x)\cos(\beta x) + r_2 \sin(\beta x)]$$

is a solution to (15).

Example: We solve

$$y'' - 4y' + 4y = x^2 + 2e^{2x}. \tag{19}$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

with one root $\lambda = 2$ whose multiplicity is 2. The fundamental system is $F.S. = \{e^{2x}, xe^{2x}\}$.

Let find a particular solution. The right hand side is not in the special form, however, we may split it into two part and first we solve

$$y'' - 4y' + 4y = x^2. \tag{20}$$

This is a special right hand side (it is enough to take $\alpha = 0$, $\beta = 0$ and $q_1 = x^2$ in the previous Theorem). As a consequence, one of the solution should be $y_p = r_1(x)$ where $r_1(x) = ax^2 + bx + c$ is a second-order polynomial. We have $y_p' = 2ax + b$ and $y_p'' = 2a$. We plug this into (20) to deduce that the particular solution is of the form

$$y_p = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}.$$

Next, we solve

$$y'' - 4y' + 4y = 2e^{2x} \tag{21}$$

Once again, we deal with the special right hand side (this time we take $\alpha = 2$, $\beta = 0$ and $q_1 = 2$). This time $\lambda = 2$ is a root of the characteristic equation and its multiplicity is 2. As a result, we are looking for solution in a form $y_r = ax^2e^{2x}$. As a result, we get $a = 1$ and thus $y_r = x^2e^{2x}$.

Thus, all solutions to (19) are of the form

$$y = x^2e^{2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8} + c_1e^{2x} + c_2xe^{2x},\ c_1, c_2 \in \mathbb{R}.$$

What if the right hand side is not in a special form? We need to use the variation of constants. This works similarly as in the case of first-order linear equations. Namely, we obtain a solution to the homogeneous problem of the form

$$y = c_1f_1 + c_2f_2 + \ldots + c_nf_n$$

where $f_1, \ldots, f_n$ are functions from the fundamental system and $c_1, \ldots, c_n$ are arbitrary real constant. Then we assume that the particular solution is of the form

$$y = c_1f_1 + c_2f_2 + \ldots + c_nf_n$$

but this time $c_1, \dots, c_n$ are unknown functions. The rest is just a matter of correct handling.

Example: We solve

$$y''' - y' = \frac{e^x}{1+e^x}.$$

The characteristic equation is

$$\lambda^3 - \lambda = 0$$

. Its roots are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$ and all solutions to the homogeneous problem are

$$y = c_1 + c_2 e^x + c_3 e^{-x},\ c_1, c_2, c_3 \in \mathbb{R}.$$

One particular solution is of the form $y = c_1(x) + c_2(x)e^x + c_3(x)e^{-x}$. We have

$$y' = c_1'(x) + c_2'(x)e^x + c_2(x)e^x + c_3'(x)e^{-x} - c_3(x)e^{-x}$$

Since there is some degree of freedom we assume that $c_1'(x) + c_2'(x)e^x + c_3'(x)e^{-x}$. Thus $y' = c_2(x)e^x - c_3(x)e^{-x}$ and

$$y''(x) = c_2'(x)e^x + c_2(x)e^x - c_3'(x)e^{-x} + c_3(x)e^{-x}.$$

Once again, we may assume $c_2'(x)e^x - c_3'(x)e^{-x} = 0$ and $y''(x) = c_2(x)e^x + c_3(x)e^{-x}$. This gives

$$y'''(x) = c_2'(x)e^x + c_2(x)e^x + c_3'(x)e^{-x} - c_3(x)e^{-x}.$$

We plug this into the given equation to get

$$c_2'(x)e^x + c_3'(x)e^{-x} = \frac{e^x}{1+e^x}$$

This gives (to be done…).

5.5 Exercises

- Try to write down a differential equation which describe a number $N(t)$ of rabbits at time t if we assume that higher number of rabbits means means higher increase.
- Of course the growth of the population of rabbits from the previous exercises cannot go forever as they will run out of available foot. Try to adjust the equation from the previous task if we assume, moreover, that the maximum population that the food can support is k.
- Use separation of variables to solve $y' = \sin(x + y)$. (Of course, one has to use some clever substitution).
- Show that the operator L defined by (9) is linear (it satisfies (10)).

6 Difference equations

6.1 Linear difference equations with constant coefficients

This subsection is devoted to the study of difference equations. Namely, we are looking for an unknown sequence $\{y(n)\}_{n=1}^{\infty}$ which fulfills

$$y(n+k) + p^1 y(n+k-1) + \ldots + p^k y(n) = a_n \tag{22}$$

where a_n is some given right hand side and $p^1, \ldots, p^k \in \mathbb{R}$ are given coefficients. Such equation is called 'linear difference equation of order k'.

Example: Assume that we have to pay a mortgage 200 000 USD. The interest of this mortgage is 0.1% per month and we pay monthly 1 000 USD. Let denote the sum we owe in the $n-$th month by $y(n)$. Clearly, $y(0) = 200\ 000$. Clearly,

$$y(n+1) = 1.001y(n) - 1\ 000.$$

This might be rewritten as

$$y(n+1) - 1.01y(n) = -1\ 000.$$

Note that the left hand side of (22) is a linear operator. We proceed similarly as in the case of the linear differential equations with constant coefficients. First of all, we find all solutions to the homogeneous case

$$y(n+k) + p^1 y(n+k-1) + \ldots + p^k y(n) = 0$$

and then we find one particular solution to non-homogeneous equation. The sum of these two outcomes gives the set of all solutions to the given problem.

The assumed solution to the homogeneous problem is $y(n) = \lambda^n$. Thus, the characteristic equation is

$$\lambda^k + p^1\lambda^{k-1} + p^2\lambda^{k-2} + \ldots + p^k = 0.$$

Theorem 6.1. *Let $\{\lambda_j = \mu_j + i\xi_j\}_{j=1}^k$ are roots of the characteristic equation of multiplicity ν^k. Then the fundamental system is*

$$\{n^\alpha \mu_j^n \sin\left(\xi_j n\frac{\pi}{2}\right), n^\alpha \mu_j^n \cos\left(\xi_j n\frac{\pi}{2}\right),\ j \in \{1, \ldots k\},\ \alpha \in \{1, \ldots, \nu_j - 1\}\}.$$

Let go back to the mortgage example. The appropriate homogeneous equation is

$$\lambda - 1.001 = 0$$

which yields $\lambda = 1.001$ and thus the fundamental system is $\{1.001^n\}$. All solutions to this homogeneous problem are of the form $y(n) = c1.001^n$ where $c \in \mathbb{R}$ is an arbitrary constant.

Special right hand side: Let $P(n)$ be a polynomial. One solution $y(n)$ of equation

$$L(y) = \alpha^n P(n)$$

is of the form

$$y(n) = n^m \alpha^n Q(n)$$

where $m = 0$ if α is not a root of the characteristic equation and m equals the multiplicity of the root α otherwise, and $Q(n)$ is a polynomial of degree at most $\deg P(n)$.

Finally, we are able to conclude the mortgage example. We need to find one solution to

$$y(n+1) - 1.001y(n) = -1\ 000.$$

The right hand side is of the special form, namely $\alpha \equiv 1$ and $P(n) = 1\ 000$ is a polynomial of degree 0. As a result, one of the solution is of the form

$$y(n) = Q(n)$$

where $Q(n) = a \in \mathbb{R}$ since it can be only 0 degree polynomial. Thus we obtain

$$a - 1.001a = -1\ 000$$

which yields $a = 1\ 000\ 000$. All solutions are of the form

$$y(n) = 1\ 000\ 000 + c1.001^n$$

and since $y(0) = 200\ 000$ we deduce

$$y(n) = 1\ 000\ 000 - 800\ 0001.001^n.$$

6.2 Recurrence relations

Usually, sequences are given by explicit formula, for example a sequence $a_n = \frac{1}{n}$ is sequence whose first few members are $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.
Sequences might be given also by recurrence relation. For example:

$$a_{n+1} = a_n + n + 1, \quad a_1 = 1.$$

How to get an explicit formula from the recurrence relation? By guessing. First, we try to guess the correct answer and then we verify our guess by induction. Recall that induction is a way how to prove a claim of the form $\forall n,\ V(n)$ and it consists of two steps:

- First we show $V(1)$.
- Next we show that $V(n) \Rightarrow V(n+1)$ for all $n \in \mathbb{N}$.

Let go back to

$$a_{n+1} = a_n + n + 1, \quad a_1 = 1.$$

The first few elements of this sequence are

$$1,\ 3,\ 6,\ 10,\ 15, \ldots$$

We may deduce that the explicit formula might be

$$a_n = \binom{n+1}{2}.$$

Now it is enough to show that such defined a_n satisfies the given recurrence relation.
First, we have

$$a_1 = \binom{2}{2}.$$

Next, we need to show that if $a_n = \binom{n+1}{2}$, then a_{n+1} defined as $a_n + n + 1$ satisfies $a_{n+1} = \binom{n+2}{2}$. But we have

$$a_n + n + 1 = \binom{n+1}{2} + \binom{n+1}{1} = \binom{n+2}{2} = a_{n+1}.$$

where we used the relation

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Thus we have just verified that the sequence fulfilling

$$a_{n+1} = a_n + n + 1, \quad a_1 = 1$$

is the sequence

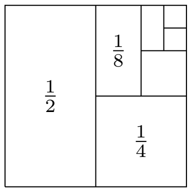
$$a_n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

6.3 Exercises

- Try to find the relation for the $n-$th of the famous Fibonacci sequence ($y(n+2) = y(n+1) + y(n)$, $y(0) = 1$, $y(1) = 1$).

7 Series

Can be a sum of infinitely many positive number finite?



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Recall, let $q \in \mathbb{R}$. Then

$$\begin{aligned} (q+1)(q-1) &= q^2 - 1 \\ (q^2+q+1)(q-1) &= q^3 - 1 \\ \left(q^n + q^{n-1} + q^{n-2} + \dots + q + 1\right)(q-1) &= q^{n+1} - 1 \end{aligned}$$

We may infer that for $q \neq 1$ it holds that

$$q^n + q^{n-1} + q^{n-2} + \dots + q + 1 = \frac{q^{n+1} - 1}{q - 1} \quad \left(= \frac{1 - q^{n+1}}{1 - q} \right)$$

which might be reformulated as

$$\sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1 - q}.$$

We may proceed to a limit with n. Assume $q \in (-1, 1)$. Then $\lim_{n\to\infty} q^{n+1} = 0$ and we infer that

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}.$$

Thus,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - 1 = \frac{1}{1 - \frac{1}{2}} - 1 = \frac{1}{\frac{1}{2}} - 1 = 1.$$

Or, equivalently,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)=\frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i}=\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right)=1.$$

Definition 7.1. *Let* $\{a_i\}_{i=0}^{\infty}\subset\mathbb{R}$ *be a sequence. We define the* $n-$th partial sum

$$s_n=\sum_{i=0}^{n}a_i.$$

If $\lim_{n\to\infty}s_n$ *exists and is finite, than we say that* $\sum_{i=0}^{\infty}a_i$ converges *and its value is* $\lim_{n\to\infty}s_n$*. If a sum does not converge, we say that it* diverges*.*

Observation 7.1. *Let* $\sum_{i=0}^{\infty}a_i$ *converges, then* $\lim_{n\to\infty}a_n=0$*.*

Proof. It holds that

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}s_n-s_{n-1}=0$$

where the last equality is true because of the arithmetic of limits. □

The last observation is quite intuitive. It states that if a sum of infinitely many numbers is finite then necessarily, these numbers have to converge to zero.
Or, on the other hand, if a sequence of numbers has a nonzero limits, then their sum cannot be finite.
Is this condition sufficient? Is it true that

$$\lim_{n\to\infty}a_n=0\Rightarrow\sum_{i=0}^{\infty}a_n<\infty?$$

Consider $\sum_{i=1}^{\infty}\frac{1}{i}$. We have

$$\left(\frac{1}{i+1}+\frac{1}{i+2}+\ldots+\frac{1}{i^2}\right)>(i^2-i)\frac{1}{i^2}=1-\frac{1}{i}$$

and therefore

$$\frac{1}{i}+\frac{1}{i+1}+\frac{1}{i+2}+\ldots+\frac{1}{i^2}>1.$$

Thus, we may split the sum into infinitely many (finite) subsums each giving a number higher than one. Therefore,

$$\sum_{i=1}^{\infty}\frac{1}{i}=\infty$$

despite the fact that $\lim_{i\to\infty}\frac{1}{i}=0$.
Roughly speaking: If a_n tends to zero sufficiently fast, $\sum_{n=0}^{\infty}a_n$ converges. What does it mean sufficiently fast and how we verify that?

7.1 Series of positive numbers

Throughout this subsection, we assume that $a_n>0$ for every $n\in\{0,1,2,3,\ldots\}$.
Comparison criterion: Let $\{a_n\}_{n=0}^{\infty}\subset\mathbb{R}$ and $\{b_n\}_{n=0}^{\infty}\subset\mathbb{R}$ fulfill $a_n\leq b_n$ for every $n\in\{0,1,2,3,\ldots\}$. Then

- if $\sum_{n=0}^{\infty} b_n$ converges, then also $\sum_{n=0}^{\infty} a_n$ converges,
- if $\sum_{n=0}^{\infty} a_n$ diverges, then also $\sum_{n=0}^{\infty} b_n$ diverges.

Example Does
$$\sum_{n=0}^{\infty} \frac{2^n + n}{5^n}$$
converge or diverge?
Since $n \leq 2^n$, we deduce
$$\frac{2^n + n}{5^n} \leq \frac{2^n + 2^n}{5^n} = 2\frac{2^n}{5^n}$$
and since
$$\sum_{n=0}^{\infty} 2\frac{2^n}{5^n} = 2\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = 2\frac{1}{1-\frac{2}{5}} < \infty,$$
we get the convergence of the given series.

Comparison scales:

- It holds that
$$\sum_{n=0}^{\infty} q^n$$
converges for $q \in (0,1)$ and diverges for $q \geq 1$.
- It holds that
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
converges for $p > 1$ and diverges for $p \leq 1$.

Example Does
$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$
converge or diverge?
First, we deduce that $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and we have $\frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2\sqrt{n+1}}$. Further
$$\sum_{n=1}^{\infty} \frac{1}{2}\frac{1}{\sqrt{n+1}} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$$
where the last series diverge. Therefore, we found a divergent series consisting of numbers lower than the original series and we infer, that the given series diverges.

The d'Alambert criterion (ration test): Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be a sequence of positive real numbers. Then

- if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges,
- if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Remark 7.1. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ *then the ration test is insufficient as it cannot decide whether the series converges or not.*

Example Let examine

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}.$$

(First, recall that $n!$ denotes a factorial of n which is defined as follows:$0! = 1$, $n! = n(n-1)!$.) We use the ration test with $a_n = \frac{(n!)^2}{(2n)!}$. We have

$$\lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(n!)^2} \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} (n+1)^2 \frac{1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4}.$$

Since $\frac{1}{4}$ is strictly less than 1, the given sum is finite due to the ration test.

The Cauchy criterion (root test): Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be a sequence of positive real numbers. Then

- if $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges,
- if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Remark 7.2. *If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ then the root test is insufficient as it cannot decide whether the series converges or not.*

Example Examine a sum

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}.$$

We use the root test. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{(n-1)} = \lim_{n\to\infty} \left(1 - \frac{2}{n+1}\right)^{(n-1)}$$

$$= e^{-2} < 1.$$

Therefore, the root test yields the convergence of the given sum.

7.2 Series of numbers with arbitrary sign

From now on we will consider sums $\sum_{n=0}^{\infty} a_n$ where, apriori, there is no assumption on the sign of a_n.

Definition 7.2. *Let*

$$\sum_{n=0}^{\infty} |a_n|$$

converges. Then we say that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent *(or converges absolutely)*

Observation 7.2. *Let $\sum_{n=0}^{\infty} a_n$ converge absolutely. Then it converges.*

Example Does a sum

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

converge or diverge?
First, let examine the absolute convergence of the series. Consider a sum

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}.$$

We have

$$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we obtain the absolute convergence of the given sum. Therefore, the given sum converges.

The Leibnitz criterion Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be a sequence of positive numbers such that

- $\lim_{n \to 0} a_n = 0$.

- a_n is a monotone sequence.

Then,

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.
Example Consider the sum

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}.$$

In order to use the Leibnitz criterion, we have to verify two assumptions. First of all

$$\lim_{n \to \infty} \frac{\sqrt{n}}{n+100} = \lim_{n \to \infty} \frac{1/\sqrt{n}}{1+\frac{100}{n}} = 0$$

and the first assumption is true.
Next, let show that the sequence is monotone (i.e. decreasing). We have to verify that $a_{n+1} < a_n$. Since the members of the sequence are positive, we can instead verify that $a_{n+1}^2 < a_n^2$. We have

$$\frac{n}{n^2+200n+10000} > \frac{n+1}{n^2+202n+10201}$$
$$n^3 + 202n^2 + 10201n > n^3 + 201n^2 + 10200n + 10000$$
$$n^2 + n > 10000.$$

and we see, that starting from, say $n = 100$, the demanded inequality is true and the sequence is decreasing. Since the finite number of terms does not matter, we may deduce that

$$\sum_{n=100}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$$

converges but this in turn implies that

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$$

converges as well.

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