UCT Mathematics

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Once, these lecture notes will contain mathematical knowledge needed to pass through math exam at the University of Chemistry and Technology. They are released online and they are available for free. On the other hand, my work on this text is still not finished and thus it may contain some mistake. In case you find any, let me know.

Contents

1 Linear Algebra

1.1 Vector spaces

Definition 1.1. *A set V endowed with operations $+$ (sum) and $.$ (multiplication by a real number) which satisfy $u+v \in V$ for all $u, v \in V$ and $\alpha.u \in V$ for all $u \in V$ and $\alpha \in \mathbb{R}$ is called* vector space *(or a linear space) if the following properties are true:*

i) $u + v = v + u$ *for all* $u, v \in V$,

ii) $u + (v + w) = (u + v) + w$ *for all* $u, w \in V$,

iii) $\exists 0 \in V$ *for which it holds that* $0 + v = v$ *for all* v,

iv) *for all* v *there is an element* $-v$ *such that* $v + (-v) = 0$,

v) $\alpha.(\beta.v) = (\alpha.\beta).v$ *for all* $\alpha, \beta \in \mathbb{R}$ *and for all* $v \in V$,

vi) $1.v = v$ *for all* $v \in V$,

vii) $(\alpha + \beta).v = \alpha.v + \beta.v$ *for all* $\alpha, \beta \in \mathbb{R}$ *and for all* $v \in V$,

viii) $\alpha.(v + w) = \alpha.v + \alpha.w$ *for all* $\alpha \in \mathbb{R}$ *and for all* $v, w \in V$.

An element of the vector space is called vector.

Remark 1.1 (on notation)**.** *It is customary to denote vectors either by bold letters (i.e., $\mathbf{v} \in V$) or by letters with an arrow (i.e., $\vec{v} \in V$). Hereinafter we use non-bold and non-arrowed letters to denote vectors (i.e., $v \in V$). This does not cause any misunderstandings. In case we work with a group of vectors $v_i \in \mathbb{R}^n$, $i \in \{1, \ldots, d\}$ and we need to highlight the $k-th$ component, we use $(v_i)_k$.*

Examples:

- The space of ordered pairs of real numbers $(u, v) \in \mathbb{R}^2$ with summation and product defined as
$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \quad \alpha(u_1, v_1) = (\alpha u_1, \alpha v_1)$$
for all $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ is a vector space.

- In general, all ordered $n-$tuples of real numbers $(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ for $n \in \mathbb{N}$ form a vector space.

- The set S of all $(x, y) \in \mathbb{R}^2$ satisfying
$$x + 2y = 0 \tag{1}$$
is a vector space. Since this is a subset of the vector space mentioned above, it is enough to verify that $((x_1, y_1), (x_2, y_2) \in S) \Rightarrow (x_1 + x_2, y_1 + y_2) \in S$ and $(\alpha \in \mathbb{R} \& (x, y) \in S) \Rightarrow (\alpha x, \alpha y) \in S$. So let (x_1, y_1) and (x_2, y_2) satisfy (1). Then $(x_1 + x_2, y_1 + y_2)$ also satisfies (1) since
$$x_1 + x_2 + 2(y_1 + y_2) = x_1 + 2y_1 + x_2 + 2y_2 = 0.$$
Next, let $\alpha \in \mathbb{R}$ be arbitrary number and let (x, y) satisfies (1). Then
$$\alpha x + 2\alpha y = \alpha(x + 2y) = 0$$
and $(\alpha x, \alpha y) \in S$.

- On the other hand, the set S of all pairs $(x, y) \in \mathbb{R}^2$ satisfying

$$x + 2y = 1$$

is not a vector space. For example, a zero vector $(0, 0)$ does not belong to S and the third property from the definition of vector space is not fulfilled.

- The set of polynomials is a vector space.
- The set of polynomials of degree 2 is not a vector space. In particular, a zero polynomial does not belong to this set as the zero polynomial has not degree 2.
- On the other hand, the set of polynomials of degree 0, 1 or 2 is a vector space.

Definition 1.2. *Let V be a vector space and let $S \subset V$ be such that*

i) $\forall s_1, s_2 \in S$*,* $s_1 + s_2 \in S$ *and*

ii) $\forall \alpha \in \mathbb{R}$ *and* $\forall s \in S$ *we have* $\alpha s \in S$*.*

Then S itself is a vector space and we say that S is a subspace *of V. If S is nonempty and $S \neq V$ then we will say that S is a* proper subspace.

Examples:

- A subset $S = \{(x, y, 0) \in \mathbb{R}^3\}$ of $V = \mathbb{R}^3$ is a proper subspace.
- All (x, y) solving $x + 2y = 0$ form a subspace of $V = \mathbb{R}^2$ see also one of the previous examples.

Definition 1.3. *Let V be a vector space, $n \in \mathbb{N}$ and $\{u_i\}_{i=1}^n \subset V$. Their* linear combination *is any vector w of the form*

$$w = \sum_{i=1}^{n} \alpha_i u_i$$

where α_i are real numbers.

Examples:

- Consider a vector space $\mathbb{R}^3$. The vector $(2, 5, 3)$ is a linear combination of $(1, 1, 0)$ and $(0, 1, 1)$ because

$$(2, 5, 3) = 2(1, 1, 0) + 3(0, 1, 1).$$

- On the other hand, $(0, -2, 1)$ is not a linear combination of $(1, 1, 0)$ and $(0, 1, 1)$. Indeed, if it was, then there would be two numbers α and β such that

$$(0, -2, 1) = \alpha(1, 1, 0) + \beta(0, 1, 1).$$

This equation can be rewritten as a system

$$\begin{aligned} 0 &= \alpha \\ -2 &= \alpha + \beta \\ 1 &= \beta \end{aligned}$$

and we deduce that it is impossible to find α and β such that these equations are fulfilled.

Definition 1.4. *The set of all linear combinations of* $v_1,\ v_2, \ldots,\ v_n$ *is called a* linear span *of a set* $\{v_1, v_2, \ldots, v_n\}$. *Precisely,*

$$\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \left\{ \sum_{i=1}^{n} \alpha_i v_i,\ \alpha_i \in \mathbb{R} \right\}.$$

Lemma 1.1. *Linear span is a vector space.*

Examples:

- The set $\{(x, y, z) \in \mathbb{R}^3,\ 2x + y + z = 0\}$ contains a span of $v_1 = (1, -2, 0)$ and $v_2 = (0, 1, 1)$ (or, for example, $w_1 = (1, 0, 2)$ and $w_2 = (1, 1, 3)$).
- Exercise: try to prove that $\{(x, y, z) \in \mathbb{R}^3,\ 2x + y + z = 0\} = \operatorname{span}\{(1, -2, 0),\ (0, 1, 1)\}$.

Definition 1.5. *Vectors* $v_1, v_2, \ldots, v_n \in V$ *are said to be* linearly dependent *if the equation*

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$$

has a nontrivial solution (i.e. a solution $\alpha_1,\ \alpha_2, \ldots,\ \alpha_n$ *where at least one coefficient is zero).* *Vectors* $v_1, v_2, \ldots, v_n \in V$ *are* linearly independent *if the equation*

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$$

has only solution $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Examples:

- Vectors $(1, 0)$, $(0, 1)$ and $(-2, 3)$ are linearly dependent since
$$2.(1, 0) + (-3).(0, 1) + 1.(-2, 3) = (0, 0).$$
- Vectors $(1, 1, 0)$, $(2, 2, 0)$ and $(-1, 0, 1)$ are linearly dependent since
$$16.(1, 1, 0) + (-8).(2, 2, 0) + 0.(-1, 0, 1) = (0, 0, 0).$$
- Vectors $(2, 3, 1, 0)$, $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$ are linearly independent. Indeed, the equation
$$\alpha(2, 3, 1, 0) + \beta(1, 0, -1, 0) + \gamma(0, 1, 0, -1) = (0, 0, 0, 0)$$
necessarily yields $\alpha = \beta = \gamma = 0$.

Definition 1.6. *Let* $V = \operatorname{span}\{v_1, v_2, \ldots, v_n\}$. *Then we say that* $\{v_1, v_2, \ldots, v_n\}$ generates V *and the vectors* $\{v_1, v_2, \ldots, v_n\}$ *are* generators *of* V.

Observation 1.1. *Let* $v_1, v_2, \ldots, v_n$ *be linearly dependent. Then one of the vectors is a linear combination of the remaining vectors. Precisely, there is* $i \in \{1, \ldots, n\}$ *such that* $v_i \in \operatorname{span}\{\{v_1, v_2, \ldots, v_n\} \setminus \{v_i\}\}$.

Proof. According to assumptions, there is $i \in \{1, \ldots, n\}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$$

has a solution with $\alpha_i \neq 0$. Assume, without lost of generality, that $i = 1$. We may rearrange the equation as

$$v_1 = -\frac{\alpha_2}{\alpha_1} v_2 - \frac{\alpha_3}{\alpha_1} v_3 - \ldots - \frac{\alpha_n}{\alpha_1} v_n.$$

□

Corollary 1.1. *Let $v_1 \in \operatorname{span}\{v_2, \ldots, v_n\}$. Then*

$$\operatorname{span}\{v_2, \ldots, v_n\} = \operatorname{span}\{v_1, v_2, \ldots, v_n\}.$$

Proof. Clearly, $\operatorname{span}\{v_2, \ldots, v_n\} \subset \operatorname{span}\{v_1, v_2, \ldots, v_n\}$. Next, let

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

Since $v_1 = \sum_{i=2}^{n} \beta_i v_i$ for some $\beta_i \in \mathbb{R}$, we get

$$v = \sum_{i=2}^{n} (\alpha_i + \alpha_1 \beta_i) v_i$$

and $v \in \operatorname{span}\{v_2, \ldots, v_n\}$. □

Definition 1.7. *Let $\{v_1, \ldots, v_n\}$ be a set of linearly independent vectors that generates V. Then $\{v_1, \ldots, v_n\}$ is a* basis *of V.*

Theorem 1.1. *Every two basis of a vector space V has the same number of elements.*

Definition 1.8. *We say that V is of dimension $n \in \mathbb{N}$ iff every basis has n elements.*

Examples:

- The set $\{(1,0),(0,1)\} \subset \mathbb{R}^2$ is a basis. Indeed, every vector $(a,b) \in \mathbb{R}^2$ can be written as $a(1,0)+b(0,1)$. Moreover, the vectors are linearly independent since $\alpha_1(1,0)+\alpha_2(0,1) = 0$ has only the trivial solution. Thus, the dimension of $\mathbb{R}^2$ is 2.

- Vectors $\{1, x, x^2\}$ form a basis of a vector space containing polynomials of degree at most two. The dimension of this vector space is thus 3.

Definition 1.9. *Let $\{v_i, i = 1, \ldots, n\}$ be independent vectors and let $v \in span\{v_i, i = 1, \ldots, n\}$. Then the numbers $\alpha_i,\ i = 1, \ldots, n$ satisfying*

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

are determined uniquely and they are called **coordinates** *of v with respect to the given basis.*

Examples

- The coordinates of $(0,1)$ with respect to $(3,2)$ and $(4,3)$ are $(-4,3)$. Indeed, $-4(3,2) + 3(4,3) = (0,1)$.

- The coordinates of $P(x) = x^2 + 3x + 4$ with respect to $Q(x) = x^2 + 2$ and $R(x) = \frac{3}{2}x + 1$ are $(1,2)$.

1.2 Matrices

Definition 1.10. *A* matrix *is a table of numbers arranged in rows and columns. Namely, let m, n be natural numbers. Then*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})_{i=1,j=1}^{m,n}$$

The matrix A has $m-$rows and $n-$columns. The matrix A is said to be of type (m, n).

Example A matrix

$$\begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & -1 \end{pmatrix}$$

has two rows and three columns and it is of type $(2, 3)$ (or it is of type two by three).

Operations with matrices Let $A = (a_{ij})_{i=1,j=1}^{m,n}$ and $B = (b_{ij})_{i=1,j=1}^{m,n}$ be two matrices of the same type. Then we define

$$A + B = (a_{ij} + b_{ij})_{i=1,j=1}^{m,n}.$$

Let $\alpha \in \mathbb{R}$. Then $\alpha A = (\alpha a_{ij})_{i=1,j=1}^{m,n}$.
For a matrix $A = (a_{ij})_{i=1,j=1}^{m,n}$ we define a transpose matrix A^T as

$$A^T = (a_{ji})_{j=1,i=1}^{n,m}$$

Let A be of type (m, n) and B be of type (n, p). Then $C := AB$ of type (m, p) is defined as

$$C = (c_{ij})_{i=1,j=1}^{m,p}$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Example

- $$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 & -5 \\ 1 & 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 & -5 \\ 1 & 1 & -2 & 2 \end{pmatrix}.$$

- $$3 \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & \frac{3}{2} \\ 6 & 6 \\ -9 & 3 \end{pmatrix}$$

- $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  or

  $$\begin{pmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 3 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix}3 & -1 & -1 & 0\end{pmatrix}^T = \begin{pmatrix}3\\-1\\-1\\0\end{pmatrix}$$

- $$\begin{pmatrix}-1 & 1 & 0\\ 2 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\-1 & -1\\0 & 1\end{pmatrix} = \begin{pmatrix}-3 & 0\\4 & -1\end{pmatrix}.$$

Remark 1.2. *Matrices of a given type (m, n) forms a vector space of dimension nm.*

Remark 1.3. *Warning:*

$$AB \neq BA.$$

Definition 1.11. *A matric A is called* **symmetric** *if $A = A^T$.*

Definition 1.12. *A* rank of matrix *A is a dimension of vector space generated by its rows. It is denoted by $rankA$.*

Observation 1.2. *It holds that $rankA = rankA^T$.*

Definition 1.13. *An* **elementary transformation** *of a matrix is*

- *scaling the entire row with a nonzero real number or*
- *interchanging two rows within a matrix or*
- *adding $\alpha-$multiple of one row to another for an arbitrary $\alpha \in \mathbb{R}$.*

Let A arise from B by one or more elementary transformations. Then we write $A \sim B$.

Example

$$\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix} \sim \begin{pmatrix}-1 & 2\\2 & 1\end{pmatrix} \sim \begin{pmatrix}-2 & 4\\2 & 1\end{pmatrix} \sim \begin{pmatrix}-2 & 4\\6 & -7\end{pmatrix}.$$

Definition 1.14. *A* leading coefficient *of a row is the first non-zero coefficient in that row. We say that a matrix A is in an* echelon form *if the leading coefficient (also called a* pivot*) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.*

Example Consider the following matrices:

$$A = \begin{pmatrix}-1 & -1 & 3 & 0\\0 & 0 & 2 & 1\\0 & 0 & 0 & -1\\0 & 0 & 0 & 0\end{pmatrix} \qquad B = \begin{pmatrix}-1 & -1 & 3 & 0\\0 & 2 & 2 & 1\\1 & 0 & -1 & -1\\0 & 0 & 0 & 3\end{pmatrix}$$

The matrix A is in echelon form whereas the matrix B is not in echelon form.

Observation 1.3. *Let A be in echelon form. Then its rank is equal to the number of non-zero rows.*

Proof. Let $v_1, \ldots, v_n$ denote the non-zero rows. It suffices to show that these vectors are linearly independent. Let solve the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0. \tag{2}$$

Let $p_1 \in \mathbb{N}$ be the position of the leading coefficient of v_1. Then the above equation yields

$$\alpha_1 (v_1)_{p_1} = 0$$

and $\alpha_1 = 0$. Therefore, the equation is simplified to

$$\alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_n v_n = 0.$$

Similarly as above, let $p_2 \in \mathbb{N}$ be the position of the leading coefficient of v_2. Then we deduce

$$\alpha_2 (v_2)_{p_2} = 0$$

and $\alpha_2 = 0$. The same can be deduced for every α_i, $i \in \mathbb{N}$ and, consequently, there is only a trivial solution to (2) $\square$

The Gauss elimination method

The Gauss elimination method is a sequence of elementary transformations which transform a given matrix A into an echelon form. As an example, we take a matrix

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 4 & 1 & 0 \\ 5 & 2 & -1 \end{pmatrix}.$$

In the first step, we use elementary transformations in order to get rid of 4 in the second row and 5 in the last row. So we add (-1) times the first row to the second and $-5/2$ times the first row to the last one. We get

$$A \sim \begin{pmatrix} 2 & 2 & -2 \\ 0 & -3 & 4 \\ 0 & -3 & 4 \end{pmatrix}.$$

Next, we want to eliminate the second element in the last row. In order to do so, we add (-1) times the second row to the last one to get

$$\begin{pmatrix} 2 & 2 & -2 \\ 0 & -3 & 4 \\ 0 & -3 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & -2 \\ 0 & -3 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & -2 \\ 0 & -3 & 4 \end{pmatrix}.$$

Here we use the fact that the zero row can be omitted without any serious consequence.

Notice that A has a rank two and that means that the vectors $(2, 2, -2)$, $(4, 1, 0)$ and $(5, 2, -1)$ are linearly dependent.

1.3 Systems of linear equations

Systems of equations

We are going to deal with system of m linear equations with n unknowns $x_1,\ x_2, \ldots,\ x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

We use notation $x = (x_1, x_2, \ldots, x_n)$, $b = (b_1, b_2, \ldots, b_n)$ and $A = (a_{ij})_{i=1,j=1}^{mn}$. Then the above system may be rewritten as

$$Ax^T = b^T.$$

The system of equations will be represented by an augmented matrix – i.e. a matrix $(A|b^T)$ where $A = (a_{i,j})_{i=1,j=1}^{mn}$ and b^T is the column on the right hand side. For example, a system of equations

$$\begin{aligned} 2x + 5y &= 10 \\ 3x + 4y &= 24 \end{aligned}$$

is represented by an augmented matrix

$$\begin{pmatrix} 2 & 5 & | & 10 \\ 3 & 4 & | & 24 \end{pmatrix}.$$

Such matrix consists of two parts – matrix $A = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$ and a vector of right hand side $b = (10, 24)$. Let solve the system by Gauss elimination:

$$\begin{pmatrix} 2 & 5 & | & 10 \\ 3 & 4 & | & 24 \end{pmatrix} \sim \begin{pmatrix} 6 & 15 & | & 30 \\ 3 & 4 & | & 24 \end{pmatrix} \sim \begin{pmatrix} 6 & 15 & | & 30 \\ 6 & 8 & | & 48 \end{pmatrix} \sim \begin{pmatrix} 6 & 15 & | & 30 \\ 0 & -7 & | & 18 \end{pmatrix}$$

The last row of the last matrix represent an equation

$$-7y = 18 \;\Rightarrow\; y = -\frac{18}{7}.$$

The first row of the last matrix represent

$$6x + 15y = 30$$

and once we plug there $y = -\frac{18}{7}$ we deduce

$$x = \frac{80}{7}.$$

Theorem 1.2 (Frobenius)**.** *A system of linear equations has solution if and only if* $rank A = rank(A|b^T)$.

Example: Solve

$$\begin{aligned} -x + y + z &= 0 \\ 2y + x + z &= 1 \\ 2z + 3y &= 2. \end{aligned}$$

We have

$$\begin{pmatrix} -1 & 1 & 1 & | & 0 \\ 1 & 2 & 1 & | & 1 \\ 0 & 3 & 2 & | & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & 2 & | & 1 \\ 0 & 3 & 2 & | & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & 2 & | & 1 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

and, according to the Frobenius theorem, there is no solution to the given system. Let us emphasize that the last row represents an equation

$$0x + 0y + 0z = 1.$$

Example Let find all solutions to the system

$$2x + y - z = 3$$
$$x - 2y + 3z = -1$$

We use the Gauss elimination in order to deduce

$$\begin{pmatrix} 2 & 1 & -1 & | & 3 \\ 1 & -2 & 3 & | & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & | & -1 \\ 2 & 1 & -1 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & | & -1 \\ 0 & 5 & -7 & | & 5 \end{pmatrix}$$

The red terms are the leading terms. The corresponding unknowns should be expressed by others. The unknown which does not have a corresponding leading term should be chosen as a parameter. Here we take $z = t$ where $t \in \mathbb{R}$ is a parameter. The last row of the last matrix yields $5y - 7t = 5$ and thus $y = \frac{7}{5}t + 1$. We deduce from the first row that $x = 1 - \frac{1}{5}t$. All solutions are of the form

$$(x, y, z) = (1, 1, 0) + t\left(-\frac{1}{5}, \frac{7}{5}, 1\right).$$

Exercise

- Solve

$$-x + py + pz = 1$$
$$x + y + pz = 2$$
$$px + y + 2pz = 5 - 2x$$

  where p is a real parameter.

1.4 Square matrices

Definition 1.15. *Matrices of type* (n, n) *where* $n \in \mathbb{N}$ *are called* square matrices.

Definition 1.16. *A matrix* I *of type* (n, n) *is called an* identity matrix *if* $I = (a_{ij})_{i=1,j=1}^{nn}$, $a_{ii} = 1$ *for all* $i \in \{1, \dots, n\}$ *and* $a_{ij} = 0$ *whenever* $i \neq j$.

For example,

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $n = 3$. It holds that $AI = IA = A$ for every matrix A of type (n, n).

Definition 1.17. *Let* A *by a matrix of type* (n, n)*. If there is a matrix* B *of type* (n, n) *such that*

$$AB = BA = I$$

then B *will be called an* inverse matrix *to* A *and we use notation* $B = A^{-1}$.

The Gauss elimination may be used to find A^{-1}. In particular, one has to write down an augmented matrix $(A|I)$ and use elementary transformations to get (I, B). If this is possible, then $B = A^{-1}$.

Example Find A^{-1} to $A = \begin{pmatrix} 2 & -1 \\ 3 & -3 \end{pmatrix}$:

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & -2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 0 & 3 & 3 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 0 & 1 & 1 & -\frac{2}{3} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 1 & 1 & -\frac{2}{3} \end{array}\right)$$

Consequently, $A^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix}$.

Definition 1.18. *A* square matrix *is a matrix of type (n, n) for some $n \in \mathbb{N}$.*
A square matrix A is called regular *if there is A^{-1}. Otherwise it is called* singular.

Observation 1.4. *Let A be a regular matrix. Then a system $Ax^T = b^T$ has a unique solution.*

Proof. Indeed, it suffices to apply A^{-1} from the left side on both sides of equation

$$Ax^T = b^T$$

to obtain

$$x^T = A^{-1}b^T.$$

□

Example The above proof describes another way how to solve a system of equations. Namely, we can first find A^{-1} and then $x^T = A^{-1}b^T$. Let solve the following two systems

$$\begin{aligned} 2x + y + z &= 3 \\ x + 3z &= -7 \\ 2x + y &= 1 \end{aligned}$$

and

$$\begin{aligned} 2x + y + z &= 0 \\ x + 3z &= 3 \\ 2x + y &= -1. \end{aligned}$$

Note that the matrix A of the systems (without the right hand side) is always the same. We compute A^{-1} as follows

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & -2 & 0 \\ 0 & 1 & -6 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 1 & 3 \\ 0 & 1 & 0 & 6 & -2 & -5 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

Thus, the first system has a solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 \\ 6 & -2 & -5 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix} = \begin{pmatrix} -13 \\ 27 \\ 2 \end{pmatrix}$$

and the second system has a solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 \\ 6 & -2 & -5 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

1.5 Determinant

Definition 1.19. *Let A be a square matrix of type $(1,1)$ – i.e., $A = (a)$ for some $a \in \mathbb{R}$. The* determinant *of such matrix A is* $\det A = a$.
Let $A = (a_{i,j})$ be a square matrix of type (n,n). We denote by M_{ij} the determinant of a matrix $(n-1, n-1)$ which arises from A by leaving out the $i-$th row and $j-$th column. Choose $k \in \{1, \ldots, n\}$. Then

$$\det A = (-1)^{k+1} a_{k1} M_{k1} + (-1)^{k+2} a_{k2} M_{k2} + \ldots + (-1)^{k+n} a_{kn} M_{kn} = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}.$$

Examples:
Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then $\det A = a_{11}a_{22} - a_{12}a_{21}$.
Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Observation 1.5. *Let A be a square matrix. Then*

- *if B arises from A by multiplying one row by a real number α, then* $\det B = \alpha \det A$.
- *If B arises from A by switching two rows, then* $\det B = -\det A$.
- *If B arises from A by adding $\alpha-$multiple of one row to another one, then* $\det B = \det A$.

Observation 1.6. *Let A be a square matrix having zeros under the main diagonal (i.e., $a_{ij} = 0$ for $i > j$). Then* $\det A = a_{11}a_{22}a_{33} \ldots a_{nn}$.

Example Compute $\det A$ for

$$A = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 0 & 3 & -3 & 1 \\ 2 & -3 & 0 & 2 \\ 0 & 0 & 3 & -1 \end{pmatrix}.$$

According to the rules for transformations, we have

$$\det\begin{pmatrix}-1 & 1 & 0 & 2\\ 0 & 3 & -3 & 1\\ 2 & -3 & 0 & 2\\ 0 & 0 & 3 & -1\end{pmatrix} = \det\begin{pmatrix}-1 & 1 & 0 & 2\\ 0 & 3 & -3 & 1\\ 0 & -1 & 0 & 6\\ 0 & 0 & 3 & -1\end{pmatrix}$$
$$= -\det\begin{pmatrix}-1 & 1 & 0 & 2\\ 0 & -1 & 0 & 6\\ 0 & 3 & -3 & 1\\ 0 & 0 & 3 & -1\end{pmatrix} = -\det\begin{pmatrix}-1 & 1 & 0 & 2\\ 0 & -1 & 0 & 6\\ 0 & 0 & -3 & 19\\ 0 & 0 & 3 & -1\end{pmatrix}$$
$$= -\det\begin{pmatrix}-1 & 1 & 0 & 2\\ 0 & -1 & 0 & 6\\ 0 & 0 & -3 & 19\\ 0 & 0 & 0 & 18\end{pmatrix} = 54.$$

Theorem 1.3. *Let A be $n \times n$ matrix. Statements following are equivalent:*

- $\det A = 0$.
- $Ax^T = 0$ *has a nontrivial solution.*
- A *is a singular matrix matrix.*
- $rankA = n$.
- *Rows of A are linearly dependent vectors.*
- *Columns of A are linearly dependent vectors.*

Theorem 1.4 (the Cramer rule)**.** *Consider a system $Ax^T = b^T$. Assume that A is a regular n by n matrix. Let $j \in \{1, \dots, n\}$ and denote by A_j a matrix arising from A by replacing $j-th$ column by a vector b^T. Then*
$$x_j = \frac{\det A_j}{\det A}.$$

Example We use the Cramer rule to solve

$$\begin{aligned} 3x - 2y + 4z &= 3\\ -2x + 5y + z &= 5\\ x + y - 5z &= 0 \end{aligned}$$

We have $A = \begin{pmatrix}3 & -2 & 4\\ -2 & 5 & 1\\ 1 & 1 & -5\end{pmatrix}$ and $\det A = -88$.

Further, $A_x = \begin{pmatrix}3 & -2 & 4\\ 5 & 5 & 1\\ 0 & 1 & -5\end{pmatrix}$ and $\det A_x = -108$. Consequently, $x = \frac{-108}{-88} = \frac{27}{22}$.

Next, $A_y = \begin{pmatrix}3 & 3 & 4\\ -2 & 5 & 1\\ 1 & 0 & -5\end{pmatrix}$ and $\det A_y = -122$. Consequently $y = \frac{-122}{-88} = \frac{61}{44}$.

Finally, $A_z = \begin{pmatrix}3 & -2 & 3\\ -2 & 5 & 5\\ 1 & 1 & 0\end{pmatrix}$ and $\det A_z = -46$. Consequently $z = \frac{-46}{-88} = \frac{23}{44}$.

1.6 Eigenvalues and eigenvectors

Definition 1.20. *Let A be a square matrix. We are looking for λ for which there is a nontrivial solution to*

$$Ax^T = \lambda x^T.$$

Such number λ is called eigenvalue.

This means that

$$(A - \lambda I)x^T = 0.$$

This equation has a nontrivial solution only if $A - \lambda I$ is a singular matrix. Consequently, λ is an eigenvalue if and only if

$$\det(A - \lambda I) = 0.$$

Definition 1.21. *The polynomial* $\det(A - \lambda I)$ *is called a* characteristic polynomial.

Definition 1.22. *Let λ be an eigenvalue of A. A vector v solving*

$$(A - \lambda I)v = 0$$

is called an eigenvector *corresponding to λ.*

Remark 1.4. *If v is an eigenvector then tv is also an eigenvector for all $t \in \mathbb{R}$.*
Let v and w be eigenvectors corresponding to the same eigenvalue. Then $tv + sw$ is also an eigenvector for all $t, s \in \mathbb{R}$.
Generally, let $u_i,\ i = \{1, \ldots, k\}$ be eigenvectors corresponding to λ. Then all their linear combinations are also eigenvectors corresponding to λ.
In what follows, if we say that there is only one eigenvector v, we mean that there is just one-dimensional space of eigenvectors spanned by v. If we say that there are two eigenvectors v, w, we mean that there is two-dimensional space of eigenvectors spanned by v, w. And so on.

Example Find all eigenvalues and eigenvectors to $A = \begin{pmatrix} 5 & 1 \\ 4 & 5 \end{pmatrix}$.
First, we find eigenvalues by solving

$$0 = \det\left(\begin{pmatrix} 5 & 1 \\ 4 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 5-\lambda & 1 \\ 4 & 5-\lambda \end{pmatrix}$$
$$= 25 - 10\lambda + \lambda^2 - 4 = \lambda^2 - 10\lambda + 21.$$

We obtain

$$\lambda_1 = 3,\ \lambda_2 = 7.$$

Consider first $\lambda_1 = 3$. Then we have to solve $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$. We have

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \end{pmatrix}$$

and we take $y = t$ and $x = -\frac{t}{2}$. Thus $(x, y) = t(-1/2, 1)$ and $v_1 = (-1/2, 1)$ is an eigenvector related to $\lambda = 3$.
Consider $\lambda_2 = 7$. Then

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 \end{pmatrix}.$$

and we take $y = t$ and $x = \frac{t}{2}$. Consequently, $v_2 = (1/2, 1)$ is an eigenvector related to the eigenvalue $\lambda = 7$.

Exercise: Find eigenvalues and eigenvectors to $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$.
First, we have to solve

$$0 = \det \begin{pmatrix} 10 - \lambda & -9 \\ 4 & -2 - \lambda \end{pmatrix} = \lambda^2 - 8\lambda + 16.$$

This yields the only solution $\lambda_1 = 4$. To find an eigenvector we solve

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 \end{pmatrix}.$$

Thus, $(3/2, 1)$ is an eigenvector.

Exercise: Find eigenvalues and eigenvectors to $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Solve

$$0 = \det \begin{pmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2.$$

We get $\lambda = 1$. To find eigenvalues we have to solve

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

The solutions are of the form $s(1, 0) + t(0, 1)$ for all real numbers $s, t \in \mathbb{R}$.

Definition 1.23. *A* generalized eigenvector *w corresponding to an eigenvalue λ is a vector satisfying*

$$(A - \lambda I)w^T = v^T$$

where v is an eigenvector corresponding to λ.

Lemma 1.2. *Let λ be a double root of the characteristic polynomial. Assume, moreover, that there is just one corresponding eigenvector. Then there is a generalized eigenvector corresponding to λ.*

Exercise: Consider again the matrix $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$. We already know that $\lambda = 4$ is the only eigenvalue and, consequently, the matrix $A - \lambda I$ has the form

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}$$

and the corresponding eigenvector is $\begin{pmatrix} 3 & 2 \end{pmatrix}$. We look for a vector $w = (x, y)$ solving

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} w^T = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

By the Gauss elimination

$$\begin{pmatrix} 6 & -9 & | & 3 \\ 4 & -6 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & | & 1 \end{pmatrix}$$

Here $y = t$, $t \in \mathbb{R}$ is free and we have $2x - 3t = 1$ and, therefore, $x = \frac{1}{2} - \frac{3}{2}t$. Every vector of the form $\left(\frac{1}{2} - \frac{3}{2}t, t\right)$ is the generalized eigenvector – for example a vector $(-1, 1)$.

1.7 Definiteness

Definition 1.24. *Let A be an n by n symmetric matrix. The mapping*

$$Q: \begin{array}{l} \mathbb{R}^n \to \mathbb{R} \\ v \mapsto vAv^T \end{array}$$

is called a quadratic form.

Examples:

- Quadratic form given by a matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is

$$(x, y) \mapsto \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 2xy + y^2$$

  and we write $Q(x, y) = x^2 - 2xy + y^2$.

- A matrix A associated with the quadratic form

$$Q(x, y, z) = x^2 - 3xz + y^2 - z^2$$

  is $A = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & -1 \end{pmatrix}$.

- A quadratic form given by $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ is

$$Q(x, y, z) = x^2 - y^2 - 2z^2 + 4xz + 2yz.$$

Definition 1.25. *A quadratic form Q is*

- positive-definite *if $Q(v) > 0$ for every $v \in \mathbb{R}^n \setminus \{0\}$*
- *positive-semidefinite if $Q(v) \geq 0$ for every $v \in \mathbb{R}^n$*
- negative-definite *if $Q(v) < 0$ for every $v \in \mathbb{R}^n \setminus \{0\}$*
- *negative-semidefinite if $Q(v) \leq 0$ for every $v \in \mathbb{R}^n$*
- *indefinite if there are $v_1, v_2 \in \mathbb{R}$ such that $Q(v_1) < 0 < Q(v_2)$*

Examples:

- $Q(x, y) = x^2 - 2xy + y^2$ is positive-semidefinite since $Q(x, y) = (x - y)^2 \geq 0$. Note that Q is not positive-definite as $Q(1, 1) = 0$.
- $Q(x, y) = x^2 - y^2$ is indefinite because $Q(1, 0) = 1 > 0$ and $Q(0, 1) = -1 < 0$.
- $Q(x, y) = x^2 + 2xy + 2y^2$ is positive-definite because $Q(x, y) = (x + y)^2 + y^2$ and this is always non-negative and $Q(x, y) = 0$ if and only if $x = y = 0$.

- $Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is indefinite. Indeed, $Q(x,y) = x^2 + 2xy = x(x+2y)$ and, clearly, $Q(1,0) = 1 > 0$ and $Q(1,-1) = -1 < 0$.

Definition 1.26. *The definiteness of a symmetric matrix A is inherited from the associated quadratic form.*

Theorem 1.5 (Sylvester rule)**.** *Let A be n by n matrix. Denote $D_0 = 1$, $D_1 = \det(a_{11})$, $D_2 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \dots,$ $D_n = \det A$ and assume $D_0, D_1, \dots, D_n \neq 0$. If all products $D_0 \cdot D_1,\ D_1 \cdot D_2, \dots, D_{n-1}D_n$ are positive, A is a positive-definite matrix. If all the products are negative, A is a negative-definite matrix.*

Examples:

- $Q(x,y) = x^2 + 2xy + 2y^2$ is positive-definite (we already know it). Nevertheless, let verify it by the Sylvester rule. The associated symmetric matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and we have $D_0 = 1$, $D_1 = 1$, $D_2 = 1$ and Q is indead positive-definite.
- Consider $Q(x,y) = -x^2 - y^2$. We have $Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ and, therefore, $D_0 = 1$, $D_1 = -1$, and $D_2 = 1$. Consequently, the Sylvester rule yields that Q is negative-definite.

2 Sequences and series

2.1 Sequences and their limits

Definition 2.1. *A function $a : \mathbb{N} \to \mathbb{R}$, Dom $a = \mathbb{N}$ is called a* sequence. We write a_n instead of $a(n)$. The whole function is then denoted $\{a_n\}_{n=1}^{\infty}$.

For example, $a_n = \frac{1}{n}$ is a sequence of numbers $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Sequence $b_n = 2^n$ is a sequence of numbers $\{2, 4, 8, 16, 32, \dots\}$. Note also that the first sequence can be written as $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and the second one as $\{2^n\}$.

Note that the sequence is actually a real function whose domain is a set of natural numbers. Thus, one can talk about boundedness and monotony in means of definitions from the previous semester. Nevertheless, let recall a definition of a monotonous sequence.

Definition 2.2. *A sequence a_n is called*

- increasing, *if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$,*
- decreasing, *if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$,*
- non-increasing, *if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$,*
- non-decreasing, *if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$.*

A sequence, which posses one of these properties is a monotone sequence.

Example

- Lets decide about the boundedness and monotonicity of $a_n = 1 - \frac{1}{n}$. The function is clearly bounded as $a_n \leq 1$ since we subtract a positive number from one and, simultanously, $a_n \geq 0$ since $a_n = 1 - \frac{1}{n} = \frac{n-1}{n}$ and this is a ration of two non-negative numbers (recall $n \geq 1$). Lets tackle the monotonicity. First few terms are: $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = \frac{2}{3}$ and therefore the first quess is that a_n is increasing. In what follows, we prove that $a_{n+1} \geq a_n$. We have

$$a_{n+1} = 1 - \frac{1}{n+1} \geq 1 - \frac{1}{n} = a_n$$

where we use that $n < n + 1 \Rightarrow \frac{1}{n} \geq \frac{1}{n+1} \Rightarrow -\frac{1}{n+1} \geq -\frac{1}{n}$. Therefore, the sequence is increasing.

- What are the properties of $a_n = \frac{n^2}{2^n}$? Clearly, this sequence is bounded from below by zero since it is a ratio of two positive numbers. Concerning the upper bound, this will be solved later once the notion of limits is introduced. Is this sequence monotone? We have $a_1 = \frac{1}{2}$, $a_2 = 1$, $a_3 = \frac{9}{8}$, $a_4 = 1$ and, clearly this sequence is not monotone as

$$a_2 < a_3 > a_4.$$

Definition 2.3. *Let a_n be a sequence. A number $A \in \mathbb{R}$ is called a limit of a_n if*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n > n_0,\ |a_n - A| < \varepsilon.$$

We then write $\lim a_n = A$.
A limit of a_n is $+\infty$ if

$$\forall M > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n > n_0,\ a_n > M$$

and we write $\lim a_n = +\infty$.
A limit of a_n is $-\infty$ if $\lim -a_n = +\infty$.

Observation 2.1. *Let a_n be a sequence and let $A \in \mathbb{R}^*$ be its limit. Then it is determined uniquely.*

Proof. Let there be two numbers $A, B \in \mathbb{R}$, $A \neq B$ (here we assume, for simplicity, that both numbers are real, for other cases see exercises) and let $\lim a_n = A$ and $\lim b_n = B$. Take $\varepsilon = \frac{1}{3}|A - B|$. According to definition, there exists n_0 such that $|a_n - A| < \varepsilon$ for all $n > n_0$ and there exists n_1 such that $|a_n - B| < \varepsilon$ for all $n > n_1$. Take $n > \max\{n_0, n_1\}$. Then

$$|A - B| = |A - a_n + a_n - B| \leq |A - a_n| + |B - a_n| < \varepsilon + \varepsilon < 3\varepsilon = |A - B|$$

which is of course a contradiction. □

Examples:

- Consider a sequence $a_n = \frac{1}{n}$. We claim that $\lim a_n = 0$. Indeed, let $\varepsilon > 0$ be an arbitrary number. Take $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\varepsilon}$. Then for all $n > n_0$ we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{n_0} < \varepsilon.$$

- Next, consider a sequence $a_n = n$ (i.e. a sequence $\{1, 2, 3, \ldots\}$). We claim that $\lim a_n = \infty$. To prove this, let $M > 0$ be an arbitrary number. Take a natural number n_0 such that $n_0 > M$. Then for all $n > n_0$ we have $a_n = n > n_0 > M$.

- Similarly, it holds that $\lim q^n = \infty$ whenever $q > 1$. Indeed, let $M > 0$ be an arbitrary number. Then taking $n > \log_q M$, we get $q^n > M$.

Lemma 2.1 (Arithmetic of limits)**.** *Let a_n and b_n be sequences and let $c \in \mathbb{R}$. Then*

$$\begin{aligned}\lim (a_n \pm b_n) &= \lim a_n \pm \lim b_n \\ \lim (a_n b_n) &= \lim a_n \cdot \lim b_n \\ \lim c a_n &= c \lim a_n \\ \lim \frac{a_n}{b_n} &= \frac{\lim a_n}{\lim b_n}\end{aligned}$$

assuming the right hand side has meaning.

To make the lemma complete we specify what is the 'meaning of the right hand side'. Besides the usual division by zero there are several others indefinite terms

$$\infty - \infty,\ \frac{\infty}{\infty},\ 0 \cdot \infty,\ \frac{0}{0},\ 1^{\infty},\ \infty^0,\ 0^0$$

which do not have any meaning. We also recall that $\frac{1}{\infty} = 0$.

Proof. Here we proof only a simplified version of this claim as we will assume that $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$. Further, as the proof does not differ from the one of Observation **??**, we consider only $\lim(a_n + b_n) = \lim a_n + \lim b_n$. Take $\varepsilon > 0$ arbitrarily. Since $\lim a_n = A$ and $\lim b_n = B$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - A| < \frac{1}{2}\varepsilon$ and $|b_n - B| < \frac{1}{2}\varepsilon$. Consequently,

$$|a_n + b_n - A - B| \leq |a_n - A| + |b_n - B| < \varepsilon$$

and we have just verified that $A + B$ is a limit of $a_n + b_n$. □

Examples: Let compute several limits.

- First of all, we will prove that $\lim q^n = \infty$ for $q > 1$. Thus, we have to show that for every M there is n_0 such that $q^n > M$. In this case, it is enough to take such natural number n_0 that $n_0 > \log_q M$. Then, necessarily, $q^n > q^{n_0} > q^{\log_q M} > M$ since $f(x) = q^x$ is increasing.
- Next, we compute $\lim n^2 - n$. One may try to write $\lim n^2 - n = \lim n^2 - \lim n = \infty - \infty$. However, the last term is an indefinite term and the arithmetic of limit cannot be used in such way. We will proceed as follows

$$\lim n^2 - n = \lim n^2 \left(1 - \frac{1}{n}\right) = \lim n^2 \left(1 - \lim \frac{1}{n}\right) = \infty(1 - 0) = \infty.$$

- The general rule how to compute a limit of 'rational sequence' is to divide by the highest power of n appearing in the denominator. Let demonstrate this (in both cases we use the arithmetic of limits as noted):

$$\lim \frac{n+1}{n^2+3} = \lim \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{3}{n^2}} \overset{AL}{=} \frac{0+0}{1+0} = 0,$$

$$\lim \frac{n^3 + 3n^2}{3n^3 + n^2} = \lim \frac{1 + 3\frac{1}{n}}{3 + \frac{1}{n}} \overset{AL}{=} \frac{1 + 3 \cdot 0}{3 + 0} = \frac{1}{3}.$$

- Let compute a limit $\lim q^n$ with $q \in (0,1)$. By use of the arithmetic of limits and the previous claim we compute

$$\lim q^n = \lim \left(\frac{1}{\frac{1}{q}}\right)^n \stackrel{AL}{=} \frac{1}{\lim \left(\frac{1}{q}\right)^n} = \frac{1}{\infty} = 0.$$

Observation 2.2. *Let a_n be a sequence with real (finite) limit A. Then a_n is a bounded sequence.*

Proof. Indeed, take (for instance) $\varepsilon = 1$. There exists $n_0 \in \mathbb{N}$ such that $\{a_n\}_{n>n_0}$ is bounded from above by $A+1$ and from below by $A-1$. Next, $\{a_1, a_2, \ldots, a_{n_0}\}$ is a finite set and thus it is bounded from above (say by $M \in \mathbb{R}$) and from below by $m \in \mathbb{R}$. Then, $\{a_n\}_{n=1}^{\infty}$ is bounded from above by $\max\{M, A+1\}$ and from below by $\min\{m, A-1\}$. $\square$

Theorem 2.1 (Heine)**.** *Let $c \in \mathbb{R}^*$ and let $d \in \mathbb{R}^*$. Then $\lim_{x\to c} f(x) = d$ if and only if $\lim f(x_n) = d$ for every sequence x_n such that $\lim x_n = c$.*

The Heine theorem allows to use the l'Hospital rule to compute the limits of sequences.
Example

- Let consider a sequence $\frac{n^2}{2^n}$. We have

$$\lim \frac{n^2}{2^n} \stackrel{Heine}{=} \lim_{x\to\infty} \frac{x^2}{2^x} \stackrel{l'H,\ \frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2x}{2^x \ln 2} \stackrel{l'H,\ \frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2}{2^x \ln 4} = 0.$$

Lemma 2.2 (Sandwich lemma)**.** *Let a_n, b_n, c_n be such that $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$. Assume, moreover, that $\lim a_n = \lim c_n = A \in \mathbb{R}^*$. Then $\lim b_n$ exists and $\lim b_n = A$.*

Proof. Take an arbitrary $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $|a_n - A| < \varepsilon$ and $|c_n - A| < \varepsilon$. There may appear one of the following cases:

- $A \ge c_n$. In that case, $|b_n - A| \le |a_n - A| < \varepsilon$.
- $A \le a_n$. In that case, $|b_n - A| \le |c_n - A| < \varepsilon$.
- $A \in (a_n, c_n)$. In that case, since $b_n \in [a_n, c_n]$, we have $|b_n - A| < |a_n - c_n| = |a_n - A + A - c_n| \le |a_n - A| + |b_n - B| < 2\varepsilon$.

No matter which one is true, we have $|b_n - A| < 2\varepsilon$ and A is a limit of b_n according to the definition of limit. $\square$

Definition 2.4. *Let a_n be a sequence and let $k : \mathbb{N} \mapsto \mathbb{N}$ be an increasing sequence of natural numbers. Then a_{k_n} is a* subsequence.

Observation 2.3. *Let a_n be a sequence such that $\lim a_n = A$, $A \in \mathbb{R}^*$. Then every subsequence a_{k_n} has a limit A.*

Proof. Once again, we assume for simplicity that $A \in \mathbb{R}$. For arbitrary $\varepsilon > 0$ there exists n_0 such that $|a_n - A| < \varepsilon$. However, as k_n is an increasing sequence of natural numbers, there exists $n_1 \in \mathbb{N}$ such that $k_n > n_0$ whenever $n > n_1$. That means that for ever $n > n_1$ we have $|a_{k_n} - A| < \varepsilon$. The proof is complete. $\square$

Example

- Let consider a limit $\lim(-1)^n$ (so $a_n = (-1)^n$). The sequence of the odd terms is $a_{2n+1} = (-1)^{2n+1} = -1$ and clearly $\lim a_{2n+1} = -1$. On the other hand, the sequence of the even terms is $a_{2n} = (-1)^{2n} = 1$ and thus $\lim a_{2n} = 1$. Consequently, the limit in question does not exist.

Theorem 2.2 (Heine)**.** *Let $c \in \mathbb{R}^*$ and let $d \in \mathbb{R}^*$. Then $\lim_{x\to c} f(x) = d$ if and only if $\lim f(x_n) = d$ for every sequence x_n such that $\lim x_n = c$.*

Example

- The previous theorem allows us to use the tools of the limits of functions. Let compute $\lim \left(\frac{n+1}{n-1}\right)^n$. We use first the Heine theorem to rewrite it as follows

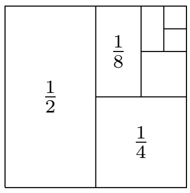
$$\lim \left(\frac{n+1}{n-1}\right)^n \overset{Heine}{=} \lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^x = \lim_{x\to\infty} e^{\ln\left(\frac{x+1}{x-1}\right)x}.$$

Due to the theorem about the limit of the composed functions, it suffices to compute the limit of the exponent. In particular

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right)x = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{\left(\frac{x+1}{x-1}\right)-1}\frac{2}{x-1}x = 2$$

and thus the limit in question is e^2. We used tools for the limits of functions (limit of composed functions, $\lim_{x\to 0}\frac{\ln(x+1)}{x} = 0$) and this was able just due to the Heine theorem.

2.2 Series



Can be a sum of infinitely many positive number finite? The picture suggests that the sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

might be finite. Lets do it precisely. First, we recall that

$$\begin{aligned} (q+1)(q-1) &= q^2 - 1 \\ (q^2+q+1)(q-1) &= q^3 - 1 \\ \left(q^n + q^{n-1} + q^{n-2} + \dots + q + 1\right)(q-1) &= q^{n+1} - 1 \end{aligned}$$

for every $q \in \mathbb{R}$. We infer that for $q \neq 1$ it holds that

$$q^n + q^{n-1} + q^{n-2} + \dots + q + 1 = \frac{q^{n+1}-1}{q-1} \quad \left(= \frac{1-q^{n+1}}{1-q}\right)$$

which might be reformulated as

$$\sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q}.$$

We proceed to a limit with n. Assume $q \in (-1,1)$. Then $\lim_{n\to\infty} q^{n+1} = 0$ and we deduce

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}.$$

Thus,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i = \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i - 1 = \frac{1}{1-\frac{1}{2}} - 1 = \frac{1}{\frac{1}{2}} - 1 = 1.$$

Or, equivalently,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right) = \frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1.$$

Definition 2.5. *Let* $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ *be a sequence. We define the* $n-$th partial sum

$$s_n = \sum_{i=0}^{n} a_i.$$

If $\lim_{n\to\infty} s_n$ *exists and is finite, than we say that* $\sum_{i=0}^{\infty} a_i$ converges *and its value is* $\lim_{n\to\infty} s_n$*. If a sum does not converge, we say that it* diverges*.*

Observation 2.4. *Let* $\sum_{i=0}^{\infty} a_i$ *converges, then* $\lim_{n\to\infty} a_n = 0$*.*

Proof. It holds that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = 0$$

where the last equality is true because of the arithmetic of limits. □

The last observation is quite intuitive. It states that if a sum of infinitely many numbers is finite then necessarily, these numbers have to converge to zero. On the other hand, numbers which do not converge to zero cannot give finite sum.

Example

- Take $a_n = 1$. The sum

$$\sum_{n=1}^{\infty} 1$$

  necessarilly diverge. Indeed, we have $s_n = n$ and $\lim s_n = \lim n = \infty$.

Is this condition sufficient? Is it true that

$$\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{i=0}^{\infty} a_n < \infty?$$

Example

- Consider $\sum_{i=1}^{\infty} \frac{1}{i}$. We have

$$\left(\frac{1}{i+1} + \frac{1}{i+2} + \ldots + \frac{1}{i^2}\right) > (i^2 - i)\frac{1}{i^2} = 1 - \frac{1}{i}$$

  and therefore

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \ldots + \frac{1}{i^2} > 1.$$

  Thus, we may split the sum into infinitely many (finite) subsums each giving a number higher than one. Therefore,

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

  despite the fact that $\lim_{i\to\infty} \frac{1}{i} = 0$.

Roughly speaking: If a_n tends to zero sufficiently fast, $\sum_{n=0}^{\infty} a_n$ converges. What does it mean sufficiently fast and how we verify that?

2.3 Series of positive numbers

Throughout this subsection, we assume that $a_n > 0$ for every $n \in \{0, 1, 2, 3, \ldots\}$.

Theorem 2.3. *Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ and $\{b_n\}_{n=0}^{\infty} \subset \mathbb{R}$ fulfill $a_n \le b_n$ for every $n \in \{0, 1, 2, 3, \ldots\}$. Then*

- *if $\sum_{n=0}^{\infty} b_n$ converges, then also $\sum_{n=0}^{\infty} a_n$ converges,*
- *if $\sum_{n=0}^{\infty} a_n$ diverges, then also $\sum_{n=0}^{\infty} b_n$ diverges.*

Example

- Does

$$\sum_{n=0}^{\infty} \frac{2^n + n}{5^n}$$

  converge or diverge?
  Since $n \le 2^n$, we deduce

$$\frac{2^n + n}{5^n} \le \frac{2^n + 2^n}{5^n} = 2\frac{2^n}{5^n}$$

  and since

$$\sum_{n=0}^{\infty} 2\frac{2^n}{5^n} = 2\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = 2\frac{1}{1-\frac{2}{5}} < \infty,$$

  we get the convergence of the given series.

To succesfully use the comparison criterion, we need to know the following scales:

- It holds that

$$\sum_{n=0}^{\infty} q^n$$

  converges for $q \in (0, 1)$ and diverges for $q \ge 1$.

- It holds that
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
converges for $p > 1$ and diverges for $p \leq 1$.

Example

- Does
$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$
converge or diverge?
First, we deduce that $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and we have $\frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2\sqrt{n+1}}$. Further
$$\sum_{n=1}^{\infty} \frac{1}{2} \frac{1}{\sqrt{n+1}} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$$
where the last series diverge. Therefore, we found a divergent series consisting of numbers lower than the original series and we infer, that the given series diverges.

The d'Alambert criterion (ration test): Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be a sequence of positive real numbers. Then

- if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges,
- if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Remark 2.1. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ *then the ration test is insufficient as it cannot decide whether the series converges or not.*

Example Let examine
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}.$$
(First, recall that $n!$ denotes a factorial of n which is defined as follows:$0! = 1$, $n! = n(n-1)!$.) We use the ration test with $a_n = \frac{(n!)^2}{(2n)!}$. We have

$$\lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(n!)^2} \frac{(2n)!}{(2n+2)!}$$
$$= \lim_{n\to\infty} (n+1)^2 \frac{1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4}.$$

Since $\frac{1}{4}$ is strictly less than 1, the given sum is finite due to the ration test.

The Cauchy criterion (root test): Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be a sequence of positive real numbers. Then

- if $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges,
- if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.

Remark 2.2. *If* $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ *then the root test is insufficient as it cannot decide whether the series converges or not.*

Example Examine a sum

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}.$$

We use the root test. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{(n-1)} = \lim_{n\to\infty} \left(1 - \frac{2}{n+1}\right)^{(n-1)} = e^{-2} < 1.$$

Therefore, the root test yields the convergence of the given sum.

2.4 Series of numbers with arbitrary sign

From now on we will consider sums $\sum_{n=0}^{\infty} a_n$ where, apriori, there is no assumption on the sign of a_n.

Definition 2.6. *Let*

$$\sum_{n=0}^{\infty} |a_n|$$

converges. Then we say that $\sum_{n=0}^{\infty} a_n$ *is* absolutely convergent *(or converges absolutely)*

Observation 2.5. *Let* $\sum_{n=0}^{\infty} a_n$ *converge absolutely. Then it converges.*

Example Does a sum

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

converge or diverge?
First, let examine the absolute convergence of the series. Consider a sum

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}.$$

We have

$$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we obtain the absolute convergence of the given sum. Therefore, the given sum converges.

The Leibnitz criterion Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be a sequence of positive numbers such that

- $\lim_{n\to 0} a_n = 0$.

- a_n is a monotone sequence.

Then,

$$\sum_{n=0}^{\infty}(-1)^n a_n$$

converges.
Example Consider the sum

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+100}.$$

In order to use the Leibnitz criterion, we have to verify two assumptions. First of all

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+100} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+\frac{100}{n}} = 0$$

and the first assumption is true.
Next, let show that the sequence is monotone (i.e. decreasing). We have to verify that $a_{n+1} < a_n$. Since the members of the sequence are positive, we can instead verify that $a_{n+1}^2 < a_n^2$. We have

$$\begin{aligned} \frac{n}{n^2+200n+10000} &> \frac{n+1}{n^2+202n+10201} \\ n^3+202n^2+10201n &> n^3+201n^2+10200n+10000 \\ n^2+n &> 10000. \end{aligned}$$

and we see, that starting from, say $n = 100$, the demanded inequality is true and the sequence is decreasing. Since the finite number of terms does not matter, we may deduce that

$$\sum_{n=100}^{\infty}(-1)^n \frac{\sqrt{n}}{n+100}$$

converges but this in turn implies that

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+100}$$

converges as well.

2.5 Gordon's growth model

Shares bought at a time $t = 0$ for P_0 give us at time $t = 1$ the following return r

$$r = \frac{Div_1 + P_1 - P_0}{P_0}$$

where Div_1 is the dividend paid during the first year. We deduce

$$P_0 = \frac{Div_1}{1+r} + \frac{P_1}{1+r}.$$

This can be used iteratively. In particular, since

$$P_1 = \frac{Div_2}{1+r} + \frac{P_2}{1+r},$$

we deduce

$$P_0 = \frac{Div_1}{1+r} + \frac{Div_2}{(1+r)^2} + \frac{P_2}{(1+r)^2}.$$

Consequently

$$P_0 = \sum_{t=1}^{\infty} \frac{Div_t}{(1+r)^t}.$$

We assume constant growth of the dividends, in particular we assume Div_1 given and $Div_t = (1+g) \cdot Div_{t-1}$. Consequently

$$P_0 = \frac{Div_1}{(} r - g)$$

2.6 Remark on exponential function

It holds that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

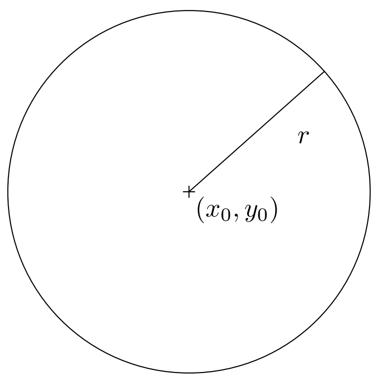
where we remind that $0! = 1$.

3 Functions of multiple variables

3.1 Few words about topology

Definition 3.1. *An open ball centered at $(x_0, y_0) \in \mathbb{R}^2$ with radius $r \in (0, \infty)$ is a set*

$$B_r(x_0, y_0) = \{(x, y) \in \mathbb{R}^2, \|(x, y) - (x_0, y_0)\| < r\}.$$



Definition 3.2. *A set $M \subset \mathbb{R}^2$ is* open *if for every $(x_0, y_0) \in \mathbb{M}$ there is $r > 0$ such that $B_r(x_0, y_0) \subset M$.*
A set M is called closed *if $\mathbb{R}^2 \setminus M$ is open.*

Example A set $M := (0, 1) \times (0, 1)$ is open. Indeed, let $(a, b) \in M$. Define $\delta = \min\{a, b, 1 - a, 1 - b\}$. Since $a \in (0, 1)$ and $b \in (0, 1)$ we have $\delta > 0$. Necessarily, $B_{\delta/2}(a, b) \subset M$. On the other hand, a set $M := [0, 1] \times (0, 1)$ is not open. Consider for example a point $(1, 1/2) \in M$.

Then every ball $B_r(1, 1/2)$ contains a point $(1 + r/2, 1/2)$ which is outside of M. Note that M is not closed. Why?

Remark 3.1.

- *$\emptyset$ and $\mathbb{R}^2$ are open sets (and closed sets as well),*
- *a union of open sets is an open set,*
- *an intersection of two open sets is an open set,*
- *a union of two closed sets is a closed set,*
- *an intersection of closed sets is a closed set.*

Observation 3.1. *Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuous function. Then $f^{-1}(A)$ is an open set for every $A \subset \mathbb{R}$ open. Similarly, $f^{-1}(B)$ is a closed set for every $B \subset \mathbb{R}$ closed.*

Question What is a *continuous function*? We will see later.
For now: A projection $p : \mathbb{R}^2 \mapsto \mathbb{R}$, $p(x, y) = x$ is a continuous function (as well as projection $q(x, y) = y$). A sum, difference and product of two continuous functions are continuous functions. A quotient of two continuous function is again a continuous function. A composition of two continuous function is a continuous function.

Example Let consider a set

$$M := \{(x, y), x \in (-1, 1),\ y < x^2\}.$$

Is this set open? First, $f(x, y) = |x|$ is a continuous function. Indeed, $f(x, y) = |p(x, y)|$ is a composition of p and $|\cdot|$. Thus, $f^{-1}((-\infty, 1)) = \{(x, y) \in \mathbb{R}^2, x \in (-1, 1)\}$ is an open set.
Next, $g(x, y) = y - x^2$ is a continuous function. Indeed, $g(x, y) = q(x, y) - p(x, y)^2$. Consequently, $f^{-1}((-\infty, 0)) = \{(x, y) \in \mathbb{R}^2, y - x^2 < 0\} = \{(x, y) \in \mathbb{R}^2, y < x^2\}$.
Since $M = f^{-1}((-\infty, 1)) \cap g^{-1}((-\infty, 0))$, we deduce that M is open.

Definition 3.3. *An* interior *of set $M \subset \mathbb{R}^2$ is a set M^0 of all points (x_0, y_0) for which there is $r > 0$ such that $B_r(x_0, y_0) \subset M$. Equivalently, it is the biggest open set contained in M.*
A closure *of a set $M \subset \mathbb{R}^2$ is a set $\overline{M}$ defined as $\overline{M} := \mathbb{R}^2 \setminus (\mathbb{R}^2 \setminus M)^0$. Equivalently, it is the smallest closed set containing M.*
A boundary *of a set M is denoted by ∂M and it is defined as $\overline{M} \setminus M^0$.*

Example Consider $M = [0, 1] \times (0, 1)$. Then $M^0 = (0, 1) \times (0, 1)$ and $\overline{M} = [0, 1] \times [0, 1]$. We deduce that

$$\partial M = \overline{M} \setminus M^0 = ([0, 1] \times \{0, 1\},\ \{0, 1\} \times [0, 1]).$$

Definition 3.4. *Let $M \subset \mathbb{R}^2$. A point $(x_0, y_0) \in \mathbb{R}^2$ is a* limit point *of M if $B_r(x_0, y_0) \cap M \neq \emptyset$ for every $r > 0$.*
A point $(x_0, y_0) \in M$ is an isolated point *of M if there is $r > 0$ such that $B_r(x_0, y_0) \cap M = \{(x_0, y_0)\}$.*

Example Consider a set $M := \{(x, y) \in \mathbb{R}, y = 0, x = 1/n,\ n \in \mathbb{N}\}$. We claim, that $(0, 0)$ is a limit point of M. Indeed, let $r > 0$. Then there is n_r such that $n_r > 1/r$ and, clearly, $(1/n_r, 0) \in M$ is such point that $\|(1/n_r, 0) - (0, 0)\| < r$ and thus $B_r(0, 0) \cap M = (1/n_r, 0)$.

3.2 Introduction to functions

Definition 3.5. *Let $M \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a nonempty set. A* real function of multiple variables *defined on a set M is a formula f which assigns a (unique) real number y to every $(x_1, x_2, \ldots, x_n) \in M$. We use the notation*

$$y = f(x_1, x_2, \ldots, x_n).$$

To denote the function itself we use a notation $f : M \mapsto \mathbb{R}$. The set M is called a domain *of f and we write $M = \text{Dom } f$.*

Remark 3.2. *In case $n = 2$ or $n = 3$ we use (x, y) or (x, y, z) instead of (x_1, x_2) or (x_1, x_2, x_3).*

Usually, the function will be given only by its formula without any specific domain. In that case, we assume that the domain is a maximal set for which has the formula sense. For example, a function

$$f(x, y) = \log(x + y)$$

is defined on a set

$$\text{Dom } f = \{(x, y) \in \mathbb{R}^2,\ x + y > 0\}.$$

Example

- Find (and sketch) a maximal set $M \subset \mathbb{R}^2$ of such pairs (x, y) for which the function

  $$f(x, y) = \frac{1}{\sqrt{x^2 + y - 1}}.$$

  Necessarily, $\sqrt{x^2 + y - 1} > 0$ and we deduce that the function has sense for all pairs satisfying

  $$x^2 + y - 1 > 0$$

  which is a part of the plane bounded by certain parabola.

Definition 3.6. *Let $z = f(x, y)$ be a function of two variables. The* graph *of f is a set*

$$\textit{graph}f = \{(x, y, f(x, y) \in \mathbb{R}^3,\ (x, y) \in \text{Dom } f\}.$$

Definition 3.7. *A* contour line *C at height $z_0 \in \mathbb{R}$ is a set*

$$\{(x, y) \in \mathbb{R}^2,\ f(x, y) = z_0\}.$$

Example

- Find contour lines at heights $z_0 = -2, -1, 0, 1, 2$ for a function

  $$f(x, y) = \frac{x^2 + y^2}{2x}.$$

  First of all, the domain of this function does not contain the y axis.
  Take $z_0 = -2$. Then $f(x, y) = -2$ yields $(x + 2)^2 + y^2 = 4$ and the contour line is the circle centered at $(-2, 0)$ with radius $r = 2$ which do not contain the origin (because of the domain of f).
  Similarly, For $z_0 = -1$ we get the circle centered at $(-1, 0)$ with radius $r = 1$ which do not contain the origin.
  The countour line at height $z_0 = 0$ is empty. For $z_0 = 1$ we get the circle centered at $(1, 0)$ with radius $r = 1$ not containing the origin.
  And finally, for $z_0 = 2$ the contour line is a circle of radius $r = 2$ with center at $(2, 0)$ with exception of the origin.

Definition 3.8. *Let $M \subset \mathbb{R}^n$ and $f : M \to \mathbb{R}$. Next, let $\varphi : I \to M$ is a curve ($I \subset \mathbb{R}$ is an interval). Then $f \circ \varphi$ is a* cross-section *of f along φ.*

Examples

- What is the graph of a function
$$f(x,y) = (x+y)^2$$
on a line $p_a : (x,y) = (a,0) + t(1,1)$, $t \in \mathbb{R}$ for some $a \in \mathbb{R}$? And how about lines $q_b : (x,y) = (b,0) + t(1,-1)$, $t \in \mathbb{R}$ for some $b \in \mathbb{R}$?
First,
$$f(a+t,t) = (a+2t)^2$$
and the graph of f along line p_a is a convex parabola with vertex in $t_0 = -\frac{a}{2}$.
Next,
$$f(b+t,-t) = (b)^2$$
and the graph is a horizontal line at height b^2.
- Lets find the graph of a cross-section
$$f(x,y) = \frac{1}{x^2+y^2}$$
along lines
$$(x,y) = t(\cos\alpha, \sin\alpha), t \in (0,\infty)$$
where $\alpha \in [0,2\pi)$ is a parameter. The function $g = f \circ \varphi$ is given as
$$g(t) = f(t\cos\alpha, t\sin\alpha) = \frac{1}{t^2(\cos^2\alpha + \sin^2\alpha)} = \frac{1}{t^2}.$$
Similarly as above, the sketch of the graph remains as an exercise for the kind reader.

Algebra of functions of two variables:
Sum, product and division is defined 'pointwisely'. Consider, for example, functions $f(x,y) = e^{xy}$ and $g(x,y) = \sqrt{1-x^2-y^2}$. Then

- $(f+g)(x,y) = e^{xy} + \sqrt{1-x^2-y^2}$,
- $(fg)(x,y) = e^{xy}\sqrt{1-x^2-y^2}$,
- $\frac{f}{g}(x,y) = \frac{e^{xy}}{\sqrt{1-x^2-y^2}}$. Beware, here we have to exclude from the domain all points where g equals zero.

Composition of functions: Let $M \subset \mathbb{R}^m$, $f : M \to \mathbb{R}^n$ (this means that there are n functions $f_i : M \to \mathbb{R}$, $i \in \{1,\dots,n\}$) and $g : \mathbb{R}^n \mapsto \mathbb{R}$. Then a *composition* is a function $h = g \circ f$ defined as
$$h(x_1,\dots,x_m) = g(f_1(x_1,\dots,x_m), f_2(x_1,\dots,x_m), \dots, f_n(x_1,\dots,x_m)).$$
Similarly, if $f : M \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ then $h = g \circ f$ is defined as $h(x_1,\dots,x_m) = g(f(x_1,\dots,x_m))$

We can also introduce the boundedness of a function $f : M \subset \mathbb{R}^n \mapsto \mathbb{R}$. This can be done similarly to the one dimensional case. The precise definition of a bounded function is left as an exercise.

3.3 Continuity

Definition 3.9. *We say that $f : M \mapsto \mathbb{R}$ is continuous at a point $(x_0, y_0) \in M$ if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ |f(x, y) - f(x_0, y_0)| < \varepsilon.$$

Let $N \subset M$ and let $f : M \mapsto \mathbb{R}$ be continuous at all points $(x_0, y_0) \in N$. Then we say that f is continuous on N. If f is continuous on Dom f *then we simply say that f is continuous.*

Observation 3.2. *Let f_1 and f_2 be continuous functions. Then*

$$f_1 + f_2,\ f_1 - f_2 \text{ and } f_1 f_2$$

are continuous function. Moreover, $\frac{f_1}{f_2}$ is a continuous function on a set $\{(x, y) \in \mathbb{R}^2, f_2(x, y) \neq 0\}$. Further, $f_1 \circ f_2$ is also a continuous function. We remind that $f(x, y) = x$ and $f(x, y) = y$ are continuous function.

Example

- A function
$$f(x, y) = \frac{x + \sqrt{x + y}}{1 + \cos^2 x}$$
wherever it is correctly defined, this means a set
$$\{(x, y) \in \mathbb{R}^2, y > -x\}.$$

3.4 Limits

Definition 3.10. *Let (x_0, y_0) be a limit point of $M \subset \mathbb{R}^2$ and let $f : M \mapsto \mathbb{R}$. We say that a limit of f at the point (x_0, y_0) is $A \in \mathbb{R}$ if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ |f(x, y) - A| < \varepsilon.$$

We write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = A$.
We say that a limit of f at the point (x_0, y_0) is ∞ if

$$\forall M > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ f(x, y) > M.$$

We write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \infty$.
We say that a limit of f at the point (x_0, y_0) is $-\infty$ if $\lim_{(x,y)\to(x_0,y_0)} -f(x, y) = -\infty$.

Observation 3.3 (Arithmetic of limits)**.** *Let f and g be two functions and let (x_0, y_0) be a limit point of* Dom f *and of* Dom g. *Then*

$$\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} (f + g)(x, y) &= \lim_{(x,y)\to(x_0,y_0)} f(x, y) + \lim_{(x,y)\to(x_0,y_0)} g(x, y) \\
\lim_{(x,y)\to(x_0,y_0)} fg(x, y) &= \lim_{(x,y)\to(x_0,y_0)} f(x, y) \lim_{(x,y)\to(x_0,y_0)} g(x, y) \\
\lim_{(x,y)\to(x_0,y_0)} \frac{f}{g}(x, y) &= \frac{\lim_{(x,y)\to(x_0,y_0)} f(x, y)}{\lim_{(x,y)\to(x_0,y_0)} g(x, y)}.
\end{aligned}$$

assuming the right hand side is well defined.

The numbers $\infty - \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are not well defined (similarly to the one dimensional case).

Observation 3.4. *A function f is continuous at point $(x_0, y_0) \in \text{Dom } f$ if and only if* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

Example

- Consider a function
$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}.$$
This function is not defined at $(0,0)$. It is possible to define the value $f(0,0)$ in such a way that f is continuous? In particular, does there exists a finite limit
$$\lim_{(x,y)\to(0,0)} f(x,y)?$$
First, we approach $(0,0)$ along the line $y = 0$. We have
$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{x\to 0} \frac{0}{x^2} = 0.$$
Next, we approach $(0,0)$ along the line $x = y$. We have
$$\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^4} = 1.$$
As a result, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Lemma 3.1 (Sandwich lemma)**.** *Let f, g, h be three functions defined on $B_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $\delta > 0$. Assume*
$$\forall (x,y) \in B_\delta(x_0,y_0) \setminus \{(x_0,y_0)\},\ g(x,y) \le f(x,y) \le h(x,y).$$
If $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = \lim_{(x,y)\to(x_0,y_0)} h(x,y) = A \in \mathbb{R}$ *then also*
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A.$$

Corollary 3.1. $\lim_{(x,y)\to(x_0,y_0)} |f(x,y)| = 0 \Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0.$

Example

- Compute
$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$
We use notation $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$. First of all, we have $\lim_{x\to 0} f(x,0) = 0$ and $\lim_{y\to 0} f(0,y) = 0$. Thus, if there is a limit, it is equal to 0. We use the well known AM-GM inequality ($2|xy| \le (x^2+y^2)$) to deduce
$$0 \le \frac{|xy|}{\sqrt{x^2+y^2}} \le \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to 0$$
as $(x,y) \to 0$. The sandwich lemma yields $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$ and we have just proven that the given limit is equal to 0.

3.5 Derivatives

Definition 3.11. *Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$ be such that $\|v\| = 1$. Let $x_0 \in M^0$. The derivative of f with respect to direction v in a point x_0 is*

$$Df(x_0, v) = g'(t)|_{t=0} \text{ where } g(t) = f(x_0 + tv)$$

Remark 3.3. *The direction of an arbitrary vector v is a unit vector $\frac{v}{\|v\|}$.*

Examples

- What is the direction of a line $p : (x, y) = (2, -1) + t(1, 3)$? The size of $(1, 3)$ is $\sqrt{1^2 + 3^2} = \sqrt{10}$. Consequently, the direction of the line is $\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$.
- Let $f(x, y) = x^2 e^y$. Let compute $Df\left((1, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right)$. The line $p(t)$ passing through $(1, 0)$ with the demanded direction has expression

  $$p(t) = \left(1 + \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right).$$

  Thus

  $$Df\left((1, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) = \left(\left(1 + \frac{t}{\sqrt{2}}\right)^2 e^{\frac{t}{\sqrt{2}}}\right)'\Bigg|_{t=0} = 1$$

Definition 3.12. *We define* partial derivatives *with respect to x_i as*

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{h \to 0} \frac{f(x_0 + he_i) - f(x_0)}{h}.$$

where e_i is the vector whose $i-th$ component is 1 and all other components are zero.

Remark 3.4. *It holds that*

$$\frac{\partial f}{\partial x}(x, y) = Df((x, y), (1, 0)), \ \frac{\partial f}{\partial y}(x, y) = Df((x, y), (0, 1))$$

whenever f is a function of two variables. Similarly, one can deduce the same rule also for a function of n variable.

Definition 3.13. *Let $x_0 \in \mathrm{Dom}\ f \subset \mathbb{R}^n$. A vector of first partial derivatives*

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0)\right)$$

is called the gradient *of f at x_0.*

Example

- Let compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for a function

  $$f(x, y) = 3x^2 y + x^2 + \log(x^2 + y^2).$$

  Let first compute $\frac{\partial f}{\partial x}$. In that case we treat y as a constant and we deduce that

  $$\frac{\partial f}{\partial x} = 6xy + 2x + \frac{2x}{x^2 + y^2}.$$

In order to compute $\frac{\partial f}{\partial y}$ we treat x as a constant and we get

$$\frac{\partial f}{\partial y} = 3x^2 + \frac{2y}{x^2+y^2}.$$

We remark that in this case we have

$$\nabla f(x,y) = \left(6xy + 2x + \frac{2x}{x^2+y^2}, 3x^2 + \frac{2y}{x^2+y^2}\right).$$

Definition 3.14. *We define second order partial derivatives as follows*

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right), \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

whenever $i, j \in \{1, \ldots, n\}$, $i \neq j$. Analogously we define the third and higher order partial derivatives. The matrix of second derivatives

$$(\nabla^2 f) = \left(\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}\right)_{i,j=1}^{n}$$

is called the Hess matrix.

Example

- Let compute the first and second order derivatives for $f(x,y) = \frac{x}{y} - e^{xy}$. We have

$$\begin{aligned}
&\frac{\partial f}{\partial x} = \frac{1}{y} - ye^{xy}, \ \frac{\partial f}{\partial y} = -\frac{x}{y^2} - xe^{xy} \\
&\frac{\partial^2 f}{\partial x^2} = -y^2 e^{xy}, \ \frac{\partial^2 f}{\partial y \partial x} = -\frac{1}{y^2} - e^{xy} - xye^{xy} \\
&\frac{\partial^2 f}{\partial y^2} = 2\frac{x}{y^3} - x^2 e^{xy}, \ \frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2} - e^{xy} - xye^{xy}.
\end{aligned}$$

  The corresponding Hess matrix is

$$\nabla^2 f = \begin{pmatrix} -y^2 e^{xy} & -\frac{1}{y^2} - e^{xy} - xye^{xy} \\ -\frac{1}{y^2} - e^{xy} - xye^{xy} & 2\frac{x}{y^3} - x^2 e^{xy} \end{pmatrix}$$

Observation 3.5. *Let the second order derivative of a function f be continuous. Then*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Theorem 3.1 (Chain rule – derivative of a composed function)**.** *Let $n = 1$ or 2 and let $f : \mathbb{R}^n \mapsto \mathbb{R}^2$ and $g : \mathbb{R}^2 \mapsto \mathbb{R}$. Then*

$$\frac{\partial (g \circ f)}{\partial x_i} = \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_i} + \frac{\partial g}{\partial y_2}\frac{\partial f_2}{\partial x_i}, \ i = \{1, n\}.$$

Example

- Let $f(x) = g(\sin x, \cos x)$. Then
$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial a}\cos x - \frac{\partial g}{\partial b}\sin x$$
where we use a notation $g = g(a, b)$.
- Let $f(x, y) = \sqrt{x^2 - y^2}$ and let $x = x(t) = e^{2t}$ and $y = e^{-t}$. Let compute $\frac{\partial f(x(t),y(t))}{\partial t}$:

$$\begin{aligned}\frac{\partial f(x(t), y(t))}{\partial t} &= \left.\frac{\partial f}{\partial x}\right|_{(x(t),y(t))}\frac{\partial x(t)}{\partial t} + \left.\frac{\partial f}{\partial y}\right|_{(x(t),y(t))}\frac{\partial y(t)}{\partial t}\\ &= \left.\frac{x}{\sqrt{x^2-y^2}}\right|_{(e^{2t},e^{-t})}2e^{2t} + \left.\frac{-y}{\sqrt{x^2-y^2}}\right|_{(e^{2t},e^{-t})}(-e^{-t}) = \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}.\end{aligned}$$

3.6 Differential

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. We try to compute an increment of a function if we move from the point (x_0, y_0) to the point $(x_0 + h, y_0 + k)$, i.e., $\Delta f(x_0, y_0) = f(x_0 + h, y_0 + k) - f(x_0, y_0)$. It can be written as

$$\Delta f(x_0, y_0) = f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) + f(x_0 + h, y_0) - f(x_0, y_0).$$

Assuming $|h|$ and $|k|$ are sufficiently small we can us an approximation

$$\begin{aligned}f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) &\sim \frac{\partial f}{\partial x}(x_0 + h, y_0)k\\ f(x_0 + h, y_0) - f(x_0, y_0) &\sim \frac{\partial f}{\partial y}(x_0, y_0)h\end{aligned}$$

Moreover, $\frac{\partial f}{\partial x}(x_0 + h, y_0) \sim \frac{\partial f}{\partial x}(x_0, y_0)$ if[1] $f \in C^1$. This yields

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) \sim \frac{\partial f}{\partial x}(x_0, y_0)h + \frac{\partial f}{\partial y}(x_0, y_0)k.$$

We denote by $\mathrm{d}x$ the change in the x coordinate and $\mathrm{d}y$ the change in the y coordinate.

Definition 3.15. *Let $f \in C^1$. Then*

$$\mathrm{d}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\mathrm{d}x + \frac{\partial f}{\partial y}(x_0, y_0)\mathrm{d}y$$

is called the differential *of f at the point (x_0, y_0).*

The differential of a function can be used to determine approximate values. Let for example determine $\sqrt{(0.03)^2 + (2.89)^2}$. Consider a function $f(x, y) = \sqrt{x^2 + y^2}$. We have $\nabla f = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$. We choose $x_0 = 0$ and $y_0 = 3$. We have $\mathrm{d}x = 0.03$ and $\mathrm{d}y = -0.11$. It holds that

$$\sqrt{(0.03)^2 + (2.89)^2} \sim \sqrt{0^2 + 3^2} + 0 \cdot 0.03 + 1 \cdot (-0.11) = 2.89$$

[1]Here $f \in C^1$ means that f has continuous first partial derivatives.

Remark 3.5. *It is worth to mention that* $\mathrm{d}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot (\mathrm{d}x, \mathrm{d}y)$. *This allows to generalize the above notion also for functions of more variables. In particular, if* $f : \mathbb{R}^n \to \mathbb{R}$, *then*[2]

$$\mathrm{d}f = \nabla f \cdot (\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n).$$

Definition 3.16. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *have continuous partial derivatives at point* (x_0, y_0). *Then the* tangent plane *of the graph of* f *at point* (x_0, y_0) *is a plane with equation*

$$z = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Example

- Let compute a tangent plane of the graph of f at point $(1, 2)$ for $f(x, y) = \sqrt{9 - x^2 - y^2}$. We have
$$\nabla f(x, y) = \left(-\frac{x}{\sqrt{9 - x^2 - y^2}}, -\frac{y}{\sqrt{9 - x^2 - y^2}} \right)$$
and $\nabla f(1, 2) = (-1/2, -1)$. Thus, the tangent plane is
$$z = 2 - 1/2(x - 1) - 1(y - 2) = 9/2 - x/2 - y.$$

3.7 The Taylor polynomial

An approximation by a differential is deduced above. In particular

$$f(x, y) \sim f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0). \tag{3}$$

Recall that we use it to compute $\sqrt{(0.03)^2 + (2.89)^2}$.
The above considerations leads to the definition of the first-order Taylor polynomial at a point (x_0, y_0) as[3]

$$T_1(x, y) = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

whenever $f : \mathbb{R}^n \to \mathbb{R}$. If f is a function of two variables then the graph of T_1 is also a tangent plane to the graph of the function f at the point (x_0, y_0) and it is the only plane which is the best approximation of the function near the point (x_0, y_0).

Definition 3.17. *Let* $f : M \subset \mathbb{R}^n \to \mathbb{R}$ *and* $x_0 \in M$. *We define the second order Taylor polynomial at a point* x_0 *as*

$$T_2(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)(\nabla^2 f(x_0))(x - x_0)^T.$$

Let us just remind that the last term is actually the quadratic form considered in Chapter 1.7

Examples

[2]And here $u \cdot v$ is a scalar multiplication of two vectors with same dimension. It can be understood as a matrix multiplication $u \cdot v^T$.

[3]As above, $\nabla f(x_0) \cdot (x - x_0)$ is a scalar product and it can be seen as a multiplication of two matrices, in particular, $\nabla f(x, y) \cdot (x - x_0)^T$.

- Let compute the second order Taylor polynomial of $f(x,y)=\sqrt{9-x^2-y^2}$ (the function from the previous exercise) at $(1,2)$. First, we have

$$\nabla^2 f(x,y)=\begin{pmatrix} \frac{-\sqrt{9-x^2-y^2}+\frac{x^2}{\sqrt{9-x^2-y^2}}}{9-x^2-y^2} & -\frac{xy}{\sqrt{9-x^2-y^2}^3} \\ -\frac{xy}{\sqrt{9-x^2-y^2}^3} & \frac{-\sqrt{9-x^2-y^2}+\frac{y^2}{\sqrt{9-x^2-y^2}}}{9-x^2-y^2} \end{pmatrix}.$$

Therefore

$$\nabla^2 f(1,2)=\begin{pmatrix} \frac{-3}{8} & -1 \\ -1 & 0 \end{pmatrix}$$

and thus

$$\begin{aligned} T_2(x) &= f(1,2)+\nabla f(1,2)\cdot(x-1,y-2)+\frac{1}{2}(x-1,y-2)(\nabla^2 f(1,2))(x-1,y-2) \\ &= 2+\left(-\frac{1}{2},-1\right)\cdot(x-1,y-2)+\frac{1}{2}(x-1,y-2)\begin{pmatrix} \frac{-3}{8} & -1 \\ -1 & 0 \end{pmatrix}(x-1,y-2) \\ &= 2-\frac{1}{2}x+\frac{1}{2}-y+2+\frac{1}{2}\left(-\frac{3}{8}(x-1)^2-2(x-1)(y-2)\right). \end{aligned}$$

- We compute an approximate value $\sqrt{(0.03)^2+(2.89)^2}$ with the help of the second order Taylor polynomial. We choose $(x_0,y_0)=(0,3)$ and we use notation $f(x,y)=\sqrt{x^2+y^2}$. We have $\frac{\partial f}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$, $\frac{\partial^2 f}{\partial x^2}=\frac{y^2}{(x^2+y^2)^{3/2}}$, $\frac{\partial^2 f}{\partial y^2}=\frac{x^2}{(x^2+y^2)^{3/2}}$, $\frac{\partial^2 f}{\partial x\partial y}=\frac{-xy}{(x^2+y^2)^{3/2}}$. We deduce that T_2 at $(0,3)$ is

$$T_2(x,y)=3+(y-3)+\frac{1}{6}x^2$$

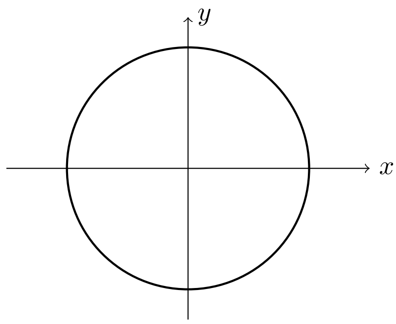
We get $T_2(0.03,2.89)=3+(-0.11)+\frac{1}{6}0.0009=2.89015$.

3.8 Implicit functions

Consider a set

$$\{(x,y)\in\mathbb{R}^2,\ x^2+y^2=1\}$$

The equation $x^2+y^2=1$ defines two function $y_1(x)$ and $y_2(x)$ where



$$y_1(x) = \sqrt{1-x^2}, \text{ Dom } y_1(x) = [-1,1],$$
$$y_2(x) = -\sqrt{1-x^2}, \text{ Dom } y_2(x).$$

What if it is impossible to express y? Consider an equation

$$f(x,y) = 0.$$

What assumptions should be imposed in order to get uniquely defined function $y(x)$?

Theorem 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$ be given. If*

i) $f \in C^k$ for some $k \in \mathbb{N}$,

ii) $f(x_0, y_0) = 0$,

iii) $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$,

Then there is a uniquely determined function $y(x)$ of class C^k on a neighborhood of point x_0 such that $f(x, y(x)) = 0$ (precisely, there is $\epsilon > 0$ and a function $y(x)$ defined on $(x_0 - \varepsilon, x_0 + \varepsilon)$ such that $f(x, y(x)) = 0$.

Example Consider an equation

$$x^3 + y^3 - 3xy - 3 = 0.$$

Is there a function $y(x)$ determined by the given equation on the neighborhood of a point $(1,2)$? According to the previous theorem, we have to verify three assumptions:
1, the function $f(x,y) = x^3 + y^3 - 3xy - 3$ should belong (at least) to C^1. That is true since $f(x,y)$ is a polynomial.
2, $f(1,2)$ should be equal to zero (or, equivalently, the given equation should be satisfied at the given point). This is also true.
3, $\frac{\partial f}{\partial y} = 3y^2 - 3x$ and therefore $\frac{\partial f}{\partial y}(1,2) = -3 \neq 0$ and the last assumption is also true.
As a result, there is a function $y(x)$ uniquely determined by the given equation in some neighborhood of point $x = 1$, $y = 2$.

Note that the last assumption in the implicit function theorem cannot be omited. Consider the first equation

$$x^2 + y^2 = 1$$

and let decide whether there is a function $y(x)$ given by that equation at the point $(1,0)$. According to the picture, it is impossible (recall the vertical line test). The theorem may not be applied. Take $f(x,y) = x^2 + y^2 - 1$. We have

$$\frac{\partial f}{\partial y} = 2y, \ \frac{\partial f}{\partial y}(1,0) = 0$$

and the third assumption is not fulfilled.
Or another example, consider a set

$$\{(x,y) \in \mathbb{R}^2, \ x^2 - y^2 = 0\}.$$

Is this set a graph of some function around a point $(0,0)$? Once again, we have $f(x,y) = x^2 - y^2$, $\frac{\partial f}{\partial y} = -2y$ and the last assumption of the implicit function theorem is not fulfilled.

Further analysis of the implicitly given function

In order to examine further qualitative properties of the given function we have to compute derivatives at the given points. The easiest method is to differentiate the given equation with respect to x (and to assume that y is in fact a function of x).

Example: Consider an equation

$$e^{2x} + e^y + x + 2y - 2 = 0.$$

This defines on a neighborhood of $(0,0)$ a function $y(x)$. Indeed, let $f(x,y) = e^{2x}+e^y+x+2y-2$. Then f is of class C^k for every $k \in \mathbb{N}$, $f(0,0) = 0$ and $\frac{\partial f}{\partial y} = e^y + 2$ which yields $\frac{\partial f}{\partial y}(0,0) = 3 \neq 0$. Let compute $y'''(0)$ (note that the third derivative exists as $f \in C^3$).

Let differentiate the equation with respect to x. We have

$$2e^{2x} + e^y y' + 1 + 2y' = 0$$

and we plug here $x = 0$ and $y = 0$ in order to get

$$2 + y'(0) + 1 + 2y'(0) = 0$$

which yields $y'(0) = -1$.

We differentiate once again with respect to x to get

$$4e^{2x} + e^y y'^2 + e^y y'' + 2y'' = 0$$

and we plug here $x = 0$, $y = 0$ and $y' = -1$. We get

$$4 + 1 + 3y''(0) = 0$$

yielding $y''(0) = -\frac{5}{3}$. We differentiate the equation for the third time in order to get

$$8e^{2x} + e^y y'^3 + e^y 2y'y'' + e^y y'y'' + e^y y''' + 2y''' = 0$$

and once again we plug there $x = 0$, $y = 0$, $y' = -1$ and $y'' = -\frac{5}{3}$. We get

$$8 - 1 + \frac{10}{3} + \frac{5}{3} + 3y''' = 0$$

which gives

$$y'''(0) = -4.$$

In particular, we may write

$$0 = \frac{\partial f(x, y(x))}{\partial x} = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{\partial y}{\partial x}$$

which gives

$$y'(x_0) = -\frac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)}.$$

3.9 Extremes

Similarly to the one-dimensional case, we talk about local and global extremes.

Definition 3.18. *Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$. We say that f attains a* local maximum *at a point $x_0 \in M^0$ if there is $r > 0$ such that $f(x_0) \geq f(x)$ for all $x \in B_r(x_0)$.*
We say that f attains a local minimum *at a point $x_0 \in M^0$ if there is $r > 0$ such that $f(x_0) \leq f(x)$ for all $x \in B_r(x_0)$.*

Definition 3.19. *Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$. We say that f attains its maximum on M at a point $x_0 \in M$ if $f(x_0) \geq f(x)$ for all $x \in M$. Similarly, f attains its minimum on M at a point $x_0 \in M$ if $f(x_0) \leq f(x)$ for all $x \in M$.*

3.9.1 Local extremes

Assume $f \in C^1$. Let f has a local extrem at (x_0, y_0). Then $g(x) = f(x, y_0)$ has also a local extreme at x_0 and, therefore, $g'(x_0) = 0$. Similarly, $h(y) = f(x_0, y)$ has a local extreme at y_0 and thus $h'(y_0) = 0$. This leads to the following observation.

Observation 3.6. *Let $f \in C^1$ have a local extreme at x_0. Then $\nabla f(x_0) = 0$.*

Definition 3.20. *A point $x_0 \in \mathrm{Dom}\, f$ such that $\nabla f(x_0) = 0$ is called a* stationary point.

How to find all local extremes of given function?
Step 1: determine the stationary point.
Step 2: examine the possible extremes in the stationary point.
Reminder: in the one-dimensional case one has to treat the sign of the second derivative in order to decide if there is an extreme in a stationary point.

Example Let find all stationary points of $f(x, y) = x^2 - y^2$. We have $\nabla f(x, y) = (2x, -2y)$ and therefore the only stationary point is $(x_0, y_0) = (0, 0)$. Is there a maximum or minimum?

Observation 3.7. *Let $f \in C^2$ and let x_0 be its stationary point. Then:*

1. *If $\nabla^2 f$ is positive definite, then f attains a local minimum at x_0,*
2. *If $\nabla^2 f$ is negative definite, then f attains a local maximum at x_0.*
3. *If $\nabla^2 f$ is indefinite, then f does not have an extreme at x_0 (saddle point).*
4. *Otherwise, we do not know anything.*

Example: Let go back to $f(x, y) = x^2 - y^2$. We already know that $(x_0, y_0) = (0, 0)$ is a stationary point. We have

$$\nabla^2 f = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus $\det \nabla^2 f(0, 0) = -4$ and there is no extreme at $(0, 0)$.

Another example Determine all local extremes of

$$f(x, y) = x^3 + 3xy^2 - 15x - 12y.$$

Step 1, stationary points:

$$\nabla f(x, y) = (3x^2 + 3y^2 - 15, 6xy - 12)$$

and we stationary points are solutions to

$$\begin{aligned} 3x^2 + 3y^2 - 15 &= 0 \\ 6xy - 12 &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} x^2 + y^2 - 5 &= 0 \\ xy &= 2. \end{aligned}$$

We deduce from the second equation that x and y are different from zero. The second equation yields $x = \frac{2}{y}$. We plug this into the first equation to deduce

$$\frac{4}{y^2} + y^2 - 5 = 0$$

which is equivalent to

$$y^4 - 5y^2 + 4 = 0.$$

We have $y^2 = 4$, $y^2 = 1$ and therefore there are four stationary points

$$A = (-1, -2),\ B = (1, 2),\ C = (2, 1),\ D = (-2, -1).$$

Step 2: We have

$$\nabla f = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix}$$

Further,

$$\nabla^2 f(A) = \begin{pmatrix} -6 & -12 \\ -12 & -6 \end{pmatrix},\ \det \nabla^2 f(A) = -108$$

and A is a saddle point.

$$\nabla^2 f(B) = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix},\ \det \nabla^2 f(B) = -108$$

and B is a saddle point.

$$\nabla^2 f(C) = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix},\ \det \nabla^2 f(C) = 108$$

and C is a point of a local minimum. The value of the local minimum is $f(C) = -28$.

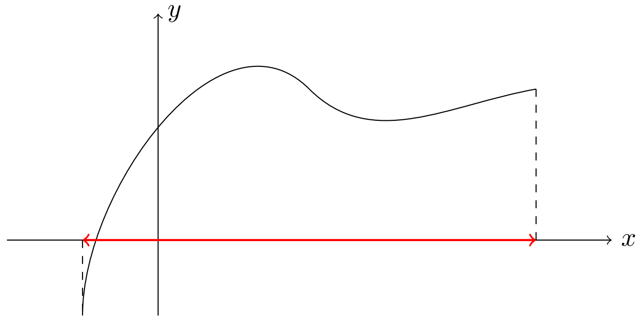
$$\nabla^2 f(D) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix},\ \det \nabla^2 f(D) = 108$$

and D is a point of a local maximum. The value of the local maximum is $f(D) = 28$.

Global extremes

Definition 3.21. *A set $M \subset \mathbb{R}^n$ is* bounded *if there is $r > 0$ such that $M \subset B_r(0)$.*

Observation 3.8. *Let $f : M \subset \mathbb{R}^n \mapsto \mathbb{R}$ be continuous. Let M be a bounded and closed set. Then there is $x_0 \in M$ where f attains its minimum on M and there is $x_1 \in M$ where M attains its maximum.*



One-dimensional case, reminder
Consider this function: One has to consider separately the interior of M and the 'boundary' of M. Although the function whose graph is in the picture has two stationary points, just one of them is a point of a global extreme. The point of the global minimum is on the edge of M.

Boundary in the two dimensional case might be a bit complicated. In order to find extremes here, we use the Lagrange multipliers method.

Theorem 3.3. *Let $f : \mathrm{Dom}\, f \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^1 and let it be defined on the neighborhood of a set M which is given as*

$$M = \{x \in \mathbb{R}^n,\ g(x) = 0\}$$

for some function $g \in C^1$. Let $\nabla g \neq 0$. If there is an extreme of f with respect to the set M then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f + \lambda \nabla g = 0.$$

Example We show how to determine a maximum and minimum of $f(x, y) = -y^2 + x^2 + \frac{4}{3}x^3$ on a set $M = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 4\}$.
The first question is whether there is a maximum and minimum. Our first claim is that the set M is closed. Why? Recall Observation 3.1. We define $g(x, y) = x^2 + y^2 - 4$ and then the set M is $g^{-1}(\{0\})$ and since $\{0\} \subset \mathbb{R}^2$ is a closed set, we deduce that M is also closed. Further, M is bounded since $M \subset B_3(0, 0)$. Therefore, according to the very first observation of this talk there has to be a maximum and minimum of f on M.
Further, it holds that $\nabla g \neq 0$ for every $(x, y) \neq (0, 0)$. Note that $(0, 0) \notin M$ and thus we may use the Lagrange multipliers. We have

$$\nabla f(x, y) = (2x + 4x^2, -2y), \quad \nabla g(x, y) = (2x, 2y).$$

We end up with a system

$$\begin{aligned} 2x + 4x^2 + 2\lambda x &= 0 \\ -2y + 2\lambda y &= 0 \\ x^2 + y^2 &= 4. \end{aligned}$$

We deduce from the second equation that $y(2\lambda - 2) = 0$ and we get that either $y = 0$ or $\lambda = 1$.
Consider first the case $y = 0$. Then the last equation yields $x^2 = 4$ and therefore $x = \pm 2$. We get two 'stationary' points

$$A = (2, 0),\ B = (-2, 0).$$

Next, assume $\lambda = 1$. The first equation then yields

$$4x + 4x^2 = 0$$

which gives $x = 0$ or $x = -1$.
Let $x = 0$. The last equation is then $y^2 = 4$ and we get $y = \pm 2$ and another two stationary points

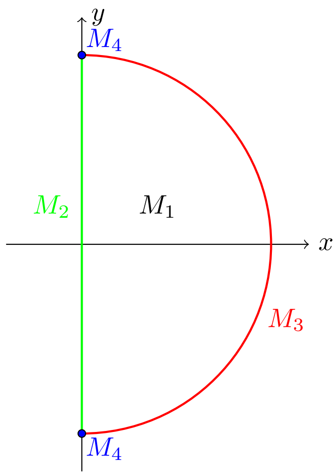
$$C = (0, 2),\ D = (0, -2).$$

Finally, let $x = -1$. Then we get $y^2 = 3$ and $y = \pm 3$ and we deduce another two stationary points

$$E = (-1, \sqrt{3}),\ D = (-1, -\sqrt{3}).$$

We have $f(A) = \frac{44}{3}$, $f(B) = -\frac{20}{3}$, $f(C) = -4$, $f(D) = -4$, $f(E) = -\frac{10}{3}$ and $f(F) = -\frac{10}{3}$. We deduce that the maximum is attained at the point $(2, 0)$ and its value is $\frac{44}{3}$, the minimum is attained at the point $(-2, 0)$ and its value is $-\frac{20}{3}$.

Example We find extremes of $f(x, y) = x^2 + y^2 - 12x + 16y$ on a set $M = \{(x, y) \subset \mathbb{R}^2,\ x^2 + y^2 \leq 25, x \geq 0\}$.



We dismantle the set into four pieces

$$\begin{aligned}
M_1 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25, x > 0\},\\
M_2 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25, x = 0\},\\
M_3 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 25, x > 0\},\\
M_4 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 25, x = 0\}.
\end{aligned}$$

and we takcle each subset separately:
- Stationary points in $M_1 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25, x > 0\}$:
We solve $\nabla f = 0$ which is

$$\begin{aligned}
2x - 12 &= 0\\
2y + 16 &= 0
\end{aligned}$$

Therefore the stationary point is $(6,-8)$. However, $6^2+(-8)^2 = 100 > 25$ and this point does not belong to M_1.
- Stationary points in $M_2 = \{(x,y) \in \mathbb{R}^2, x^2+y^2 < 25, x = 0\}$:
We are going to consider a function $f(x,y)$ on line $x = 0$. Therefore it is enough to examine function $f(0,y) =: h(y)$. We have

$$h(y) = y^2 + 16y$$

and therefore $h'(y) = 2y + 16$. The resulting stationary point is $x = 0$, $y = -8$. However, $(-8)^2 > 25$ and the point $(0,-8)$ does not belong to M_2.
- Stationary points in $M_3 = \{(x,y) \in \mathbb{R}^2, x^2+y^2 = 25, x = 0\}$:
There is a constraint $g(x,y) = x^2+y^2-25$. We have $\nabla g = (2x, 2y)$ and we have $\nabla g \neq 0$ for every $(x,y) \in M_3$. The system $\nabla f + \lambda \nabla g = 0$ complemented with $g = 0$ has form

$$\begin{aligned} 2x - 12 + 2x\lambda &= 0 \\ 2y + 16 + 2y\lambda &= 0 \\ x^2 + y^2 &= 25. \end{aligned}$$

We may deduce that $y \neq 0$ (otherwise the second equation cannot be true) and $\lambda \neq 0$ (otherwise $x = 6$, $y = -8$ and the last equation is not fulfilled. The first and second equation might be rewritten as

$$\begin{aligned} x\lambda &= 6 - x \\ y\lambda &= -8 - y \end{aligned}$$

and we divide the first equation by the second to get

$$\frac{\lambda x}{\lambda y} = \frac{6-x}{-8-y}.$$

This yields

$$\frac{x}{y} = \frac{x-6}{8+y}$$

and

$$8x + xy = xy - 6y$$

and therefore

$$y = -\frac{4}{3}x.$$

We plug this into the last equation ($x^2+y^2 = 25$) to get

$$x^2 + \frac{16}{9}x^2 = 25$$

which yields $x = \pm 3$. Therefore we have two stationary points $(-3,4)$ and $(3,-4)$, however, the first one does not belong to M_3. So we take into consideration only $A = (3,-4)$.
- Stationary points in $M_4 = \{(x,y) \in \mathbb{R}^2, x^2+y^2 = 25,\ x = 0\}$: This set consists only of two points. Indeed, let both equations holds at once. Then necessarily

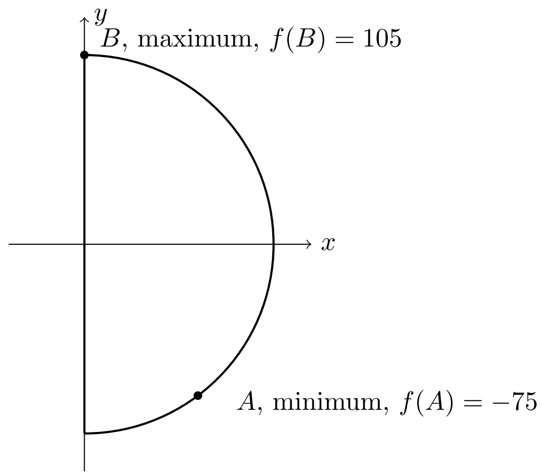
$$y^2 = 25$$

and we have two points $B = (0,5)$ and $C = (0,-5)$. These two points have to be considered as there might appear global extremes (although these points are not stationary).

- Final evaluation: We have just three points where the extremes might be attained: $A = (3, -4)$, $B = (0, 5)$ and $C = (0, -5)$. We have

$$\begin{aligned} f(A) &= -75 \\ f(B) &= 105 \\ f(C) &= -55. \end{aligned}$$

We deduce that the minimum of f on set M is attained at the point $(3, -4)$ and its value is -75, the maximum of f on set M is attained at point $(0, 5)$ and its value is 105. Next we present just



slight modification for two constraints

Theorem 3.4. *Let* $f : \mathrm{Dom}\, f \subset \mathbb{R}^n \to \mathbb{R}$ *be of class* C^1 *and let it be defined on the neighborhood of a set* M *which is given as*

$$M = \{x \in \mathbb{R}^n,\ g(x) = 0,\, h(x) = 0\}$$

for some functions $g,\, h \in C^1$. *Let* $\nabla g \neq 0$ *and* $\nabla h \neq 0$. *If there is an extreme of* f *with respect to the set* M *then there exist* $\lambda,\, \mu \in \mathbb{R}$ *such that*

$$\nabla f + \lambda \nabla g + \mu \nabla h = 0.$$

The least square method

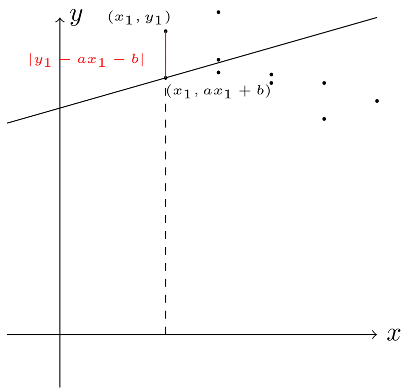
We will solve the following exercise: Assume that the cost of a car (of one given type) depends linearly on its age, i.e.,

$$y = ax + b,\ \ a, b \in \mathbb{R},$$

where y is the price of a car and x is its age.
Our aim now is to determine this function (constants a and b) from the given sets of data. Below we have a table of particular cars (their price does not follow strictly the above rule since the price come from the free market)

x	2	3	3	3	4	4	5	5	6
y	28.7	24.8	26.0	30.5	23.8	24.6	23.8	20.4	22.1

To find the line which fits best to the given data, we use the least squares method. This means that we are going to minimize the 'distance' between the line $ax + b$ and the given data. We define such distance as sum of squares:



$$|y_1 - ax_1 - b|^2 + |y_2 - ax_2 - b|^2 + \ldots + |y_n - ax_n - b|^2 = \sum_{i=1}^{n} |y_i - ax_i - b|^2.$$

This sum of squares in infact a function f of variables a and b of the form

$$f(a, b) = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

and we are going to minimize this sum of squares. We compute the partial derivative

$$\frac{\partial f}{\partial a} = -2 \sum_{i=1}^{n} (y_i - ax_i - b)x_i, \quad \frac{\partial f}{\partial b} = -2 \sum_{i=1}^{n} (y_i - ax_i - b).$$

and we deduce that the stationary point of this function has to fulfill

$$\sum_{i=1}^{n} (y_i - ax_i - b)x_i = 0$$
$$\sum_{i=1}^{n} (y_i - ax_i - b) = 0.$$

Recall that unknowns are a and b. We reformulate this into

$$\left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i$$
$$\left(\sum_{i=1}^{n} x_i\right) a + nb = \sum_{i=1}^{n} y_i.$$

Recall our example

x	2	3	3	3	4	4	5	5	6
y	28.7	24.8	26.0	30.5	23.8	24.6	23.8	20.4	22.1

where we have

$$n = 9, \ \sum_{i=1}^{9} x_i = 35, \ \sum_{i=1}^{9} x_i^2 = 149, \ \sum_{i=1}^{9} y_i = 224.7, \ \sum_{i=1}^{9} x_i y_i = 848.5.$$

We and up with equation

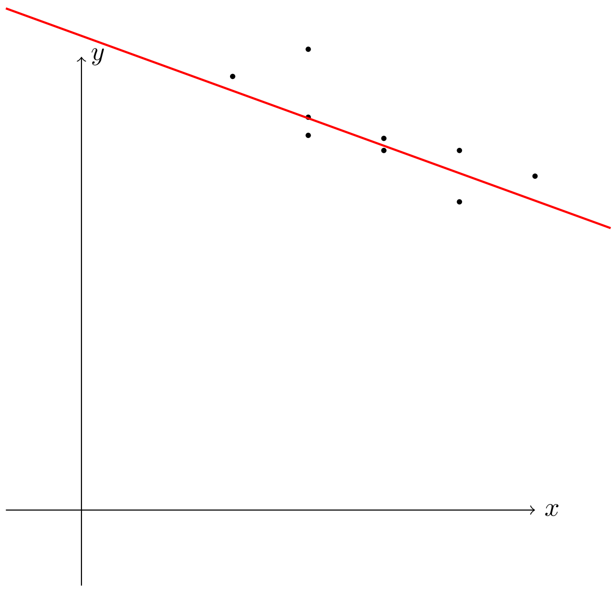
$$\begin{aligned} 149a + 35b &= 848.5 \\ 35a + 9b &= 224.7 \end{aligned}$$

which has (approximate) solution

$$a = -2.02, \quad b = 32.8.$$

Thus, the desired line has equation

$$y = -2.02x + 32.8$$



4 Systems of ODEs

4.1 Introduction

Problem:

- Two large tanks, each holding 24 liters of a brine solution, are interconnected by pipes. Fresh water flows into tank A at a rate of 6 L/min, and fluid is drained out of tank B at the same rate; also 8 L/min of fluid are pumped from tank A to tank B, and 2 L/min from tank B to tank A. The liquids inside each tank are kept well stirred so that each mixture is homogeneous. If, initially, the brine solution in tank A contains x_0 kg of salt and that in tank B initially contains y_0 kg of salt, determine the mass of salt in each tank at time $t > 0$.

Let denote:
amount of salt in the first tank: x
amount of salt in the second tank: y
salt flowing out of the first tank per one minute: $\frac{x}{24}8$
salt flowing out of the second tank per one minute: $\frac{y}{24}2 + \frac{y}{24}6$
salt flowing into the first tank per one minute: $\frac{y}{24}2$
salt flowing into the second tank per one minute: $\frac{x}{24}8$

We arrive at the system

$$\begin{aligned} x' &= -\frac{1}{3}x + \frac{1}{12}y \\ y' &= \frac{1}{3}x - \frac{1}{3}y \end{aligned}$$

which can be rewritten as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is in particular a system of first-order linear equations.

In what follows, we will tackle a system of ODEs of the form[4]

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t) \tag{4}$$

where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$ and $\mathbf{b}(t) = (b_1(t), \dots, b_n(t))^T$ are $n-$dimensional vectors and A is an n by n square matrix.

Definition 4.1. *The set of functions defined on $\mathbb{R}$ and solving* (4) *is called a* general solution*. One of this function is called a* particular solution*.*

We emphasize that higher order linear differential equations with constant coefficients might be rewritten into a system of first order linear equations. Indeed, consider

$$y'' + ky' + my = 0.$$

We denote $x = y'$ and then it holds that $x' = -kx - my$ and the above system might be rewritten as

$$\begin{aligned} x' &= -kx - my \\ y' &= x \end{aligned}$$

Theorem 4.1. *Assume A is a constant n by n matrix and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be n linearly independent solutions to the homogeneous system*

$$\mathbf{x}'(t) = A\mathbf{x}(t) \tag{5}$$

[4]Hereinafter, we use the boldface letters to denote the vector in a column form, i.e.,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

on the interval I. Then every solution to (5) *on I can be expressed in the form*

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + \ldots + c_n\mathbf{x}_n(t),$$

where $c_1, \ldots, c_n$ are real constants.

Definition 4.2. *A set of solutions $\{\mathbf{x_1}, \ldots, \mathbf{x}_n\}$ that are linearly independent is called a* fundamental solution set *for* (5).

Theorem 4.2. *If $\mathbf{x}_p$ is a particular solution to the nonhomogeneous system*

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t) \tag{6}$$

on the interval I and $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a fundamental solution set on I for the corresponding homogeneous system $\mathbf{x}'(t) = A\mathbf{x}(t)$, then every solution to (6) *on I can be expressed in the form*

$$\mathbf{x}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t),$$

where $c_1, \ldots, c_n$ are real constants.

Proof is left as an exercise for interested readers.

The above theorem yields an approach to solving linear systems of the form $\mathbf{x}' = A\mathbf{x} + b$. Namely, we have to perform the following two steps:

1. Find a fundamental solution set for the corresponding homogeneous system $\mathbf{x}' = A\mathbf{x}$.
2. Find one particular solution to the non-homogeneous system.

Then, the general solution can be written as a sum of outcomes of the previous two steps. The forthcoming

4.2 Homogeneous systems with constant coefficients

We are going to solve

$$\mathbf{x}' = A\mathbf{x} \tag{7}$$

Assume (and that is something usual in the case of linear system with constant coefficients) that the solution is of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

where $\lambda \in \mathbb{R}$ and $\mathbf{v}$ is an $n-$dimensional vector constant in t. We have

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{v}$$

and once we plug this into (7), we deduce

$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v}.$$

We may divide by $e^{\lambda t}$ to deduce

$$\lambda\mathbf{v} - A\mathbf{v} = 0.$$

The above equation has a non-trivial solution only if λ is an eigenvalue. In such case, $\mathbf{v}$ is the corresponding eigenvector.

Example

- Let try to solve the initial value problem given at the beginning of this lesson, i.e.,

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \mathbf{x}, \ \mathbf{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

  To compute the eigenvalues of $A = \begin{pmatrix} -\frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$ we have to compute a determinant

$$\det(A - \lambda I) = \det \begin{pmatrix} -\frac{1}{3} - \lambda & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{3} - \lambda \end{pmatrix} = \lambda^2 + \frac{2}{3}\lambda + \frac{1}{12}.$$

  Therefore, the eigenvalues are solutions to

$$\lambda^2 + \frac{2}{3}\lambda + \frac{1}{12} = 0.$$

  We get $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = -\frac{1}{6}$.
  Let take λ_1. Then the corresponding eigenvector whould satisfy $\begin{pmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix} \mathbf{v_1} = 0$ and this can be solved by GEM as follows

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix} \sim \begin{pmatrix} \frac{1}{6} & \frac{1}{12} \end{pmatrix}.$$

  The solution (one of many) is $\mathbf{v_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.
  Similarly, for λ_2 we have

$$\begin{pmatrix} -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix} \sim \begin{pmatrix} -\frac{1}{6} & \frac{1}{12} \end{pmatrix}$$

  and the second eigenvector is $\mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
  The set of all solution (the general solution) is

$$\mathbf{x}(t) = c_1 e^{-1/2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-1/6t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

  In order to reach the initial condition we deduce that

$$\mathbf{x}(0) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

  and the constants c_1 and c_2 has to be determined from the equation

$$\begin{aligned} -c_1 + c_2 &= x_0 \\ 2c_1 + 2c_2 &= y_0. \end{aligned}$$

What if the eigenvalues are not real? And what if the eigenvalues are not distinct? (the characteristic polynomial has a double (triple, etc.) root? That is the content of the two following sections .

4.2.1 Complex eigenvalues

Example

- Find a general solution of
$$\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \mathbf{x}.$$
To find the eigenvalues we have to solve
$$0 = \det \begin{pmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{pmatrix} = (1+\lambda)(3+\lambda) + 2 = \lambda^2 + 4\lambda + 5.$$
Therefore,
$$\begin{aligned} \lambda^2 + 4\lambda + 4 &= -1 \\ (\lambda+2)^2 &= -1 \\ \lambda + 2 &= \pm i. \end{aligned}$$
We get $\lambda_1 = -2+i$, $\lambda_2 = -2-i$. (Similarly, we may deduce that $\lambda_{1,2} = \frac{-4\pm\sqrt{-4}}{2}$). Consider λ_1. We have
$$\begin{pmatrix} 1-i & 2 \\ -1 & -1-i \end{pmatrix} \sim \begin{pmatrix} 1-i & 2 \end{pmatrix}$$
and the corresponding eigenvector is $\mathbf{v}_1 = (2, i-1)$. Here we note that $\lambda_2 = \overline{\lambda}_1$ and $\mathbf{v}_2 = \overline{\mathbf{v_1}}$ where $\overline{(\alpha+\beta i)} = \alpha - \beta i$.
We obtain that one solution is of the form
$$\mathbf{x}(t) = e^{(-2+i)t}(2, i-1) = e^{(-2+i)t}\left((2,-1) + i(0,1)\right).$$
Recall that
$$e^{a+bi} = e^a(\cos b + i \sin b).$$
Therefore, we can write
$$\begin{aligned} \mathbf{x}(t) &= e^{-2t}\left(\cos t + i \sin t\right)\left((2,-1) + i(0,1)\right) \\ &= e^{-2t}\left(\cos t(2,-1) - \sin t(0,1)\right) + ie^{-2t}\left(\sin t(2,-1) + \cos t(0,1)\right). \end{aligned}$$
The real part represents one solution, the imaginary part the second one. Thus, the general solution has a form
$$\mathbf{x}(t) = c_1 e^{-2t}\left(\cos t(2,-1) - \sin t(0,1)\right) + c_2 e^{-2t}\left(\sin t(2,-1) + \cos t(0,1)\right)$$
where c_1 and c_2 are arbitrary real constants. The same considerations lead to the following theorem.

Theorem 4.3. *If the real matrix A has complex eigenvalues $\alpha \pm \beta i$ with corresponding eigenvectors $\mathbf{a} + i\mathbf{b}$, then the two linearly independent real vector solutions to $\mathbf{x}' = A\mathbf{x}$ are*
$$\begin{aligned} &e^{\alpha t} \cos \beta t \mathbf{a} - e^{\alpha t} \sin \beta t \mathbf{b} \\ &e^{\alpha t} \sin \beta t \mathbf{a} + e^{\alpha t} \cos \beta t \mathbf{b}. \end{aligned}$$

4.2.2 Double roots

Here we distinquish two cases: either there are two linearly independent eigenvectors corresponding to one eigenvalue, or there is just one.

Examples

- Let solve
$$\mathbf{x}' = A\mathbf{x}$$
where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The characteristic equation is
$$0 = \det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2.$$
There is just one eigenvalue $\lambda = 1$ and the corresponding eigenvectors solves
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$
We deduce that there are two corresponding eigenvectors $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$ (actually, all linear combinations of these two are eigenvectors as well). Thus the generalized solution is of the form
$$\mathbf{x}(t) = c_1 e^t (1, 0) + c_2 e^t (0, 1)$$
for some $c_1, c_2 \in \mathbb{R}$.

- Consider now
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x}.$$
The characteristic equation is
$$0 = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2(3-\lambda)$$
and the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. Take $\lambda_1 = 3$. Then
$$\begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$
and the corresponding eigenvector is $(0, 2, 1)$. Thus, the fundamental solution set contains a function
$$\mathbf{x}(t) = e^{3t}(0, 2, 1).$$
Take $\lambda_2 = 1$. Then
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
and the corresponding eigenvector is $\mathbf{v}_1 = (0, 0, 1)$. Thus, the fundamental solution set contains a function
$$\mathbf{x}(t) = e^t(0, 0, 1).$$
But we need one additional function in the fundamental solution set. How to get it?

Matrix exponential
The exponential function $e^x : \mathbb{R} \mapsto \mathbb{R}$ can be defined as an infinite sum

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Similarly, let A be a square matrix. Then we write

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \ldots$$

It holds that

$$\left(e^{At}\right)' = 0 + A + A^2t + \frac{A^3t^2}{2} + \ldots = A\left(I + At + \frac{A^2t^2}{2} + \ldots\right) = Ae^{At}$$

and therefore the columns of the matrix e^{At} form the fundamental solution set of

$$\mathbf{x}(t) = A\mathbf{x}.$$

This also means that every solution is of the form $\mathbf{x}(t) = e^{At}\mathbf{v}$ where $\mathbf{v}$ is an arbitrary $n-$dimensional vector.

Let $\mathbf{v}$ be an eigenvector. Then

$$e^{At}\mathbf{v} = e^{\lambda t}e^{(A-\lambda t)}\mathbf{v} = e^{\lambda t}\left(I\mathbf{v} + t(A-\lambda I)\mathbf{v} + \frac{t^2}{2}(A-\lambda I)\mathbf{v} + \ldots\right) = e^{\lambda t}\mathbf{v}$$

Let $\mathbf{w}$ is a generalized eigenvector (recall Definition 1.23). Then

$$(A-\lambda I)^2\mathbf{w} = (A-\lambda I)(A-\lambda I)\mathbf{w} = (A-\lambda I)\mathbf{v} = 0$$

and we deduce that

$$e^{At}\mathbf{w} = e^{\lambda t}e^{(A-\lambda t)}\mathbf{w} = e^{\lambda t}\left(I\mathbf{w} + t(A-\lambda I)\mathbf{w} + \frac{t^2}{2}(A-\lambda I)\mathbf{w} + \ldots\right) = e^{\lambda t}(\mathbf{w} + t\mathbf{v}).$$

Back to our example: we have

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}\mathbf{x}.$$

We have already deduced that $\lambda_1 = 3$ has corresponding eigenvector $(0, 2, 1)$ and the double root $\lambda_2 = 1$ has a corresponding eigenvector $\mathbf{v} = (0, 0, 1)$. Now, we have to find a corresponding generalized eigenvector $\mathbf{w}$ which satisfies

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}\mathbf{w} = \mathbf{v}$$

and we use the Gauss elimination method to deduce

$$\begin{pmatrix} 0 & 0 & 0 & | & 0 \\ 1 & 2 & 0 & | & 0 \\ 0 & 1 & 0 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 1 & 0 & | & 1 \end{pmatrix}$$

Solutions are of the form $(-2,1,0)+r(0,0,1)$ for any $r \in \mathbb{R}$. It is enough to choose one solution, say $(-2,1,0)$. According to our considerations, we deduce that one solution is of the form

$$\mathbf{x}(t) = e^t((-2,1,0) + t(0,0,1)).$$

Thus, the generalized solution for the given problem is

$$\mathbf{x}(t) = c_1 e^{3t}(0,2,1) + c_2 e^t(0,0,1) + c_3 e^t((-2,1,0) + t(0,0,1))$$

for some $c_1, c_2, c_3 \in \mathbb{R}$.

To summarize:

Observation 4.1. *Let the real matrix A has an eigenvalue $\lambda \in \mathbb{R}$ which is a double root of the characteristic equation. Let there be just one corresponding eigenvector $\mathbf{v}$ and let $\mathbf{w}$ be a generalized eigenvector. Then the fundamental solution set contains the functions*

$$e^{\lambda t}\mathbf{v}, \quad e^{\lambda t}\left(\mathbf{w} + t\mathbf{v}\right).$$

4.3 Non-zero right hand side

This time we tackle the problem

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

where $\mathbf{f}(t)$ is a nonzero vector-valued function. We already know how to find all solution to the corresponding homogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

4.3.1 Undetermined coefficients

Example

- Let solve

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \mathbf{x}(t) + t \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}.$$

  First, let find all solutions to the corresponding homogeneous system. The characteristic equation is

$$0 = \det \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix} = (\lambda-3)^2(\lambda+3).$$

  For $\lambda_1 = 3$ we have

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$$

  and the corresponding eigenvectors are $\mathbf{v}_1 = (1,0,1)$ and $\mathbf{v}_2 = (0,1,1)$. For $\lambda_2 = -3$ we have

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 1 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 3 \end{pmatrix}$$

and the corresponding eigenvector is $\mathbf{v}_3 = (1, 1, -1)$. All solutions to the homogeneous problem have form

$$\mathbf{x}(t) = e^{3t}(c_1(1,0,1) + c_2(0,1,1)) + e^{-3t}c_3(1,1,-1)$$

where $c_1, c_2, c_3 \in \mathbb{R}$.
Let find one particular solution to

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \mathbf{x}(t) + t \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$$

We can assume that the solution is of the form $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are vectors constant in time. Thus we have $\mathbf{x}'(t) = \mathbf{a}$ and we plug this into the given equation in order to deduce

$$\mathbf{a} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} (\mathbf{a}t + \mathbf{b}) + t \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$$

We compare coefficients in order to deduce

$$0 = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \mathbf{a} + \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$$

and

$$\mathbf{a} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \mathbf{b}.$$

To find $\mathbf{a}$ we use the GEM as follows

$$\begin{pmatrix} 1 & -2 & 2 & | & 9 \\ -2 & 1 & 2 & | & 0 \\ 2 & 2 & 1 & | & 18 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 & | & 9 \\ 0 & -3 & 6 & | & 18 \\ 0 & 6 & -3 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 & | & 9 \\ 0 & -3 & 6 & | & 18 \\ 0 & 0 & 9 & | & 36 \end{pmatrix}$$

and we deduce that $\mathbf{a} = (5, 2, 4)$.
Next, we have

$$\begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \mathbf{b}$$

and we once again use the GEM to get

$$\begin{pmatrix} 1 & -2 & 2 & | & 5 \\ -2 & 1 & 2 & | & 2 \\ 2 & 2 & 1 & | & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 & | & 5 \\ 0 & -3 & 6 & | & 12 \\ 0 & 6 & -3 & | & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 & | & 5 \\ 0 & -3 & 6 & | & 12 \\ 0 & 0 & 9 & | & 18 \end{pmatrix}$$

and we have $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Thus, all solutions to the given equation are of the form

$$\mathbf{x}(t) = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + e^{3t} \left(c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) + c_3 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Observation 4.2. *Let* $\mathbf{f}(t) = e^{rt}t^m\mathbf{g}$ *where* g *is a constant vector. Then one solution is of the form*

$$\mathbf{x}_p(t) = e^{rt}\left(t^{m+s}\mathbf{a}_{m+s} + t^{m+s-1}\mathbf{a}_{m+s-1} + \ldots + t\mathbf{a}_1 + \mathbf{a}_0\right)$$

where $\mathbf{a}_i$ *are constant vectors and* s *is an appropriately chosen integer.*

4.4 Systems in a plane

During this subsection we consider systems of the form

$$\frac{\partial x}{\partial t} = f(x, y)$$
$$\frac{\partial y}{\partial t} = g(x, y)$$

Note that this time, the system is not necessarily linear, however, it is autonomous – the right hand side in $t-$independent.

Definition 4.3. *If* $x(t)$ *and* $y(t)$ *is a solution pair to the above mentioned system for* t *in the interval* I*, then a plot in the* xy*-plane of the parametrized curve* $(x(t), y(t))$ *for* t *in* I*, together with arrows indicating its direction with increasing* t*, is said to be a* trajectory *of the system. In such a context we call the* xy*-plane the* phase plane*.*

Example no. 1:

$$x' = -x$$
$$y' = -2y$$

Recall that

$$\frac{\partial y}{\partial x} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} = \frac{-2y}{-x}.$$

This yields $y = cx^2$ and thus we get the picture as above. Note that, since $y'(t)$ and $x'(t)$ are negative for $x, y > 0$, we get trajectories aiming to the origin. See picture above.
Example no. 2:

$$x' = x$$
$$y' = 2y$$

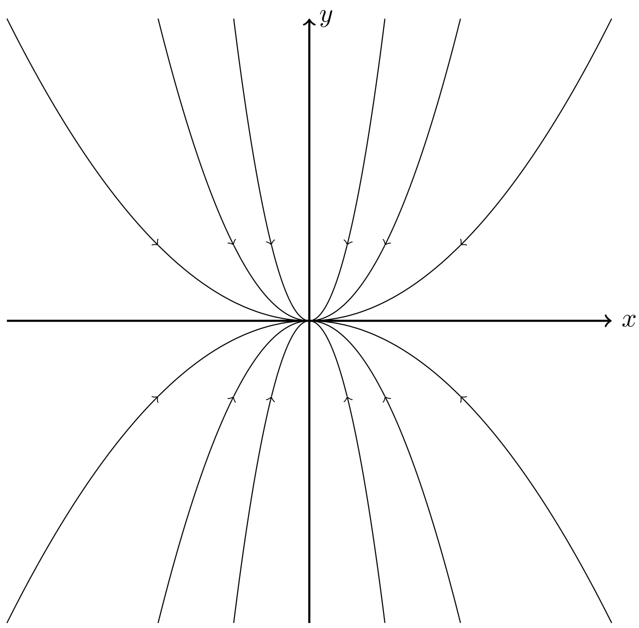
This time, the picture is the same as above with only one exception – the arrows aim away of origin (see the picture below), try to justify why.

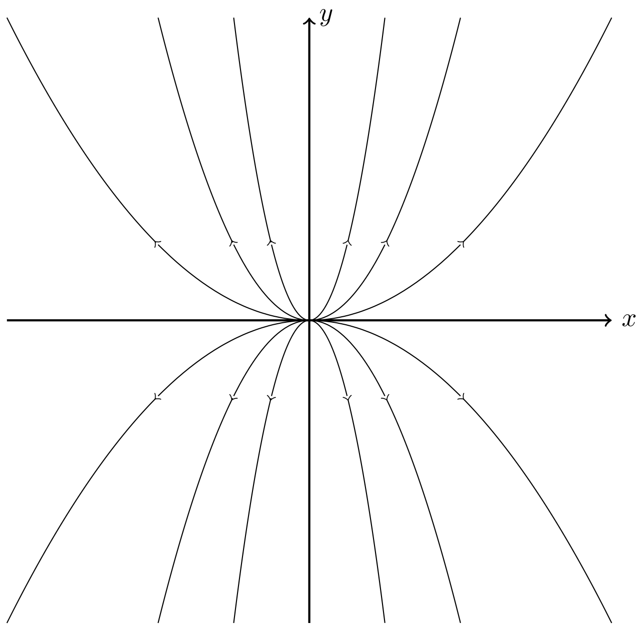
Definition 4.4. *: A point* $(x_0, y_0) \in \mathbb{R}^2$ *where* $f(x_0, y_0) = g(x_0, y_0) = 0$ *is called a* critical point *(or* equilibrium point*) of the given system. The corresponding solution* $x \equiv x_0$*,* $y \equiv y_0$ *is called an* equilibrium solution *(or* stationary solution*).*

Observation 4.3. *Let* $x(t)$ *and* $y(t)$ *be a solution on* $[0, \infty)$ *to the given system (we assume* f *and* g *are continuous). If the limits*

$$\lim_{t\to\infty} x(t) = x_0, \quad \lim_{t\to\infty} y(t) = y_0$$

exist and are finite, then (x_0, y_0) *is a critical point of the system.*





Types of equilibrium points:

- Stable node (asymptotically stable)
- Unstable node
- Stable spiral (asymptotically stable)

- Unstable spiral
- Saddle (unstable)
- Center (stable, but not asymptiotically)

Example: Find the critical points and sketch trajectories in the phase plane for

$$x' = -y(y-2)$$
$$y' = (x-2)(y-2).$$

What is the behavior of the solutions starting from $(3,0)$, $(5,0)$ and $(2,3)$?

Let consider a special case of a linear system in a plane, i.e.,

$$x' = a_{11}x + a_{12}y + b_1$$
$$y' = a_{21}x + a_{22}y + b_2$$

which might be shortened to

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

In what follows, we assume that $\mathbf{b} = 0$. This assumption will be commented later.

From what we know, we deduce that

- Let the eigenvalues λ_1 and λ_2 be real, distinct and both positive. Then $(0,0)$ is an unstable node.
- Let the eigenvalues λ_1 and λ_2 be real, distinct and both negative. Then $(0,0)$ is a stable node.
- Let the eigenvalues λ_1 and λ_2 be real and have oposite signs. Then $(0,0)$ is a saddle point.
- Let the eigenvalues λ_1 and λ_2 be equal. Then $(0,0)$ is either a proper node (stable or unstable) or an improper node (stable or unstable).
- Let the eigenvalues be complex, i.e. $\lambda_{12} = a \pm bi$ where $a, b \in \mathbb{R}$. Then $(0,0)$ is a spiral. It is a stable spiral if $a < 0$ and it is an unstable spiral if $a > 0$.

Example

- Find and classify the critical point of the linear system

$$x' = 2x + y - 3$$
$$y' = -3x - 2y - 4.$$

The critical point satisfies

$$2x + y - 3 = 0$$
$$-3x - 2y - 4 = 0$$

which yields $(x_0, y_0) = (10, -17)$. We introduce the new variables $u = x-10$ and $v = y+17$. Clearly $u' = x'$ and $v' = y'$ and thus

$$u' = 2u + v$$
$$v' = -3u - 2v.$$

and we use the theory for the homogeneous systems. The matrix

$$\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. The critical point $(10, -17)$ is the saddle point and it is an unstable equilibrium.

4.4.1 Almost linear systems

Definition 4.5. *An* almost linear system *is a system of the form*

$$x' = a_{11}x + a_{12}y + f(x,y)$$
$$y' = a_{21}x + a_{22}y + g(x,y)$$

Here we assume that f and g are just small perturbations. In particular,

$$\lim_{\sqrt{x^2+y^2}\to 0} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0, \qquad \lim_{\sqrt{x^2+y^2}\to 0} \frac{g(x,y)}{\sqrt{x^2+y^2}} = 0.$$

The system

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is called a corresponding linear system*.*

Note, that the origin is an equilibrium point of the almost linear system.

Theorem 4.4. *The stability properties of the critical point at the origin for the almost linear system are the same as the stability properties of the origin for the corresponding linear system with one exception: When the eigenvalues are pure imaginary, the stability properties for the almost linear system cannot be deduced from the corresponding linear system.*

Competing species:
The population of two species x and y (independent on each other) might be governed by the logistic equations

$$x' = k_1x(C_1 - x) = k_1C_1x - k_1x^2 = a_1x - b_1x^2$$
$$y' = k_2x(C_2 - y) = a_2y - b_2y^2$$

Now assume that both species compete for the same food. In such a case, the capacity C might be exhausted by both x and y and therefore we assume

$$x' = a_1x - b_1x^2 - c_1xy$$
$$y' = a_2y - b_2y^2 - c_2xy$$

Consider two competing species whose population is governed by

$$x' = x(7 - x - 2y) = 7x - x^2 - 2xy$$
$$y' = y(5 - y - x) = 5y - y^2 - xy.$$

There are four critical points: $A = (0, 0)$, $B = (0, 5)$, $C = (7, 0)$, and $D = (3, 2)$.

Take A. The appropriate linear system is

$$x' = 7x$$
$$y' = 5y.$$

The eigenvalues are 7 and 5 and therefore A is an unstable node.

Take B. We have to employ the change of coordinates $u = x$, $v = y - 5$. Thus the system is rewritten as

$$u' = -3u - u^2 - 2uv$$
$$v' = -5v - 5u - v^2 - uv.$$

The appropriate linear system is

$$u' = -3u$$
$$v' = -5v - 5u$$

and the appropriate eigenvalues are both negative (-3 and -5). Consequently, B is a stable node.

Let examine C. We use the change of coordinates $u = x - 7$, $v = y$. The system takes form

$$u' = -7u + 14v - u^2 - 2uv$$
$$v' = -2v - v^2 - uv.$$

The eigenvalues of the respective linear system

$$u' = -7u + 14v$$
$$v' = -2v$$

are both negative (-2 and -7) and the point C is a stable node.

Finally, we take $u = x - 3$ and $v = y - 2$ in order to handle D. The system is of the form

$$x' = -3u - 6v - u^2 - 2uv$$
$$y' = -2u - 2v - uv - v^2.$$

The appropriate linear system

$$x' = -3u - 6v$$
$$y' = -2u - 2v.$$

has two eigenvalues 1 and -6 and therefore D is a saddle.

5 Difference equations

5.1 Linear difference equations with constant coefficients

This subsection is devoted to the study of difference equations. Namely, we are looking for an unknown sequence $\{y(n)\}_{n=1}^{\infty}$ which fulfills

$$y(n+k) + p^1 y(n+k-1) + \ldots + p^k y(n) = a_n \tag{8}$$

where a_n is some given right hand side and $p^1, \ldots, p^k \in \mathbb{R}$ are given coefficients. Such equation is called 'linear difference equation of order k'.

Example: Assume that we have to pay a mortgage 200 000 USD. The interest of this mortgage is 0.1% per month and we pay monthly 1 000 USD. Let denote the sum we owe in the $n-$th month by $y(n)$. Clearly, $y(0) = 200\ 000$. Clearly,

$$y(n+1) = 1.001 y(n) - 1\ 000.$$

This might be rewritten as

$$y(n+1) - 1.01 y(n) = -1\ 000.$$

Note that the left hand side of (8) is a linear operator. We proceed similarly as in the case of the linear differential equations with constant coefficients. First of all, we find all solutions to the homogeneous case

$$y(n+k) + p^1 y(n+k-1) + \ldots + p^k y(n) = 0$$

and then we find one particular solution to non-homogeneous equation. The sum of these two outcomes gives the set of all solutions to the given problem.

The assumed solution to the homogeneous problem is $y(n) = \lambda^n$. Thus, the characteristic equation is

$$\lambda^k + p^1 \lambda^{k-1} + p^2 \lambda^{k-2} + \ldots + p^k = 0.$$

Theorem 5.1. *Let $\{\lambda_j\}_{j=1}^{k}$ are real roots of the characteristic equation of multiplicity ν_k. Then the fundamental system is*

$$\{n^\alpha \lambda_j^n,\ j \in \{1, \ldots k\},\ \alpha \in \{1, \ldots, \nu_j - 1\}\}.$$

Let go back to the mortgage example. The appropriate homogeneous equation is

$$\lambda - 1.001 = 0$$

which yields $\lambda = 1.001$ and thus the fundamental system is $\{1.001^n\}$. All solutions to this homogeneous problem are of the form $y(n) = c 1.001^n$ where $c \in \mathbb{R}$ is an arbitrary constant.

Special right hand side: Let $P(n)$ be a polynomial. One solution $y(n)$ of equation

$$L(y) = \alpha^n P(n)$$

is of the form

$$y(n) = n^m \alpha^n Q(n)$$

where $m = 0$ if α is not a root of the characteristic equation and m equals the multiplicity of the root α otherwise, and $Q(n)$ is a polynomial of degree at most $\deg P(n)$.

Finally, we are able to conclude the mortgage example. We need to find one solution to

$$y(n+1) - 1.001y(n) = -1\ 000.$$

The right hand side is of the special form, namely $\alpha \equiv 1$ and $P(n) = 1\ 000$ is a polynomial of degree 0. As a result, one of the solution is of the form

$$y(n) = Q(n)$$

where $Q(n) = a \in \mathbb{R}$ since it can be only 0 degree polynomial. Thus we obtain

$$a - 1.001a = -1\ 000$$

which yields $a = 1\ 000\ 000$. All solutions are of the form

$$y(n) = 1\ 000\ 000 + c1.001^n$$

and since $y(0) = 200\ 000$ we deduce

$$y(n) = 1\ 000\ 000 - 800\ 0001.001^n.$$

Complex roots Let solve the equation

$$y(n+2) - 2y(n+1) + 2y(n) = 0. \tag{9}$$

The characteristic polynomial is then of the form

$$\lambda^2 - 2\lambda + 2 = 0$$

and the solutions are

$$\lambda_{1,2} = 1 \pm i.$$

In this case, we follow the following theorem:

Theorem 5.2. *Let $\lambda_{1,2} = a \pm bi$ be roots of the characteristic equation. Then the appropriate functions in the fundamental system are*

$$r^n \cos(n\theta) \quad \text{and } r^n \sin(n\theta)$$

where $r = \sqrt{a^2 + b^2}$ and $\theta \in [0, 2\pi)$ is such that $a = r\cos\theta$ and $b = r\sin\theta$.

In our case, $a = 1$ and $b = 1$ and we get $r = \sqrt{2}$. Further, we deduce from

$$1 = \sqrt{2}\cos\theta, \quad 1 = \sqrt{2}\sin\theta$$

that $\theta = \frac{\pi}{4}$. All solutions to (9) are of the form

$$y(n) = c_1\sqrt{2}^n \cos\left(\frac{n\pi}{4}\right) + c_2\sqrt{2}^n \sin\left(\frac{n\pi}{4}\right), \quad c_1, c_2 \in \mathbb{R}.$$

5.2 Recurrence relations

Usually, sequences are given by explicit formula, for example a sequence $a_n = \frac{1}{n}$ is sequence whose first few members are $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.
Sequences might be given also by recurrence relation. For example:

$$a_{n+1} = a_n + n + 1, \quad a_1 = 1.$$

How to get an explicit formula from the recurrence relation? By guessing. First, we try to guess the correct answer and then we verify our guess by induction. Recall that induction is a way how to prove a claim of the form $\forall n,\ V(n)$ and it consists of two steps:

- First we show $V(1)$.
- Next we show that $V(n) \Rightarrow V(n+1)$ for all $n \in \mathbb{N}$.

Let go back to

$$a_{n+1} = a_n + n + 1, \quad a_1 = 1.$$

The first few elements of this sequence are

$$1,\ 3,\ 6,\ 10,\ 15, \ldots$$

We may deduce that the explicit formula might be

$$a_n = \binom{n+1}{2}.$$

Now it is enough to show that such defined a_n satisfies the given recurrence relation.
First, we have

$$a_1 = \binom{2}{2}.$$

Next, we need to show that if $a_n = \binom{n+1}{2}$, then a_{n+1} defined as $a_n + n + 1$ satisfies $a_{n+1} = \binom{n+2}{2}$. But we have

$$a_n + n + 1 = \binom{n+1}{2} + \binom{n+1}{1} = \binom{n+2}{2} = a_{n+1}.$$

where we used the relation

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Thus we have just verified that the sequence fulfilling

$$a_{n+1} = a_n + n + 1, \quad a_1 = 1$$

is the sequence

$$a_n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

The above example yields another way how to solve the difference equations. In particular, the given recurrence relation

$$a_{n+1} = a_n + n + 1$$

is actually a linear difference equation

$$y(n+1) - y(n) = n + 1.$$

However, the above method is applicable also to nonlinear cases – try the following exercises:
Exercises:

- Find the formula for a_n if

  $$a_n = na_{n-1}, \ a_0 = 1.$$

  Prove the correctness of your answer.

- Find the formula for a_n if

  $$a_n = a_{n-1}^2, \ a_0 = 2$$

  and prove its correctness.

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