UCT Mathematics

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Once, these lecture notes will contain mathematical knowledge needed to pass through math exam at the University of Chemistry and Technology. They are released online and they are available for free. On the other hand, my work on this text is still not finished and thus it may contain some mistake. In case you find any, let me know.

Contents

1 Functions of two and more variables – extremes **3**
1.1 Extremes . . . 3
1.2 Global extremes . . . 5
1.3 Extrema with respect to a compact set . . . 6

2 Integrals **11**
2.1 Motivation, basic notions . . . 11
2.2 Calculation – basic methods . . . 12
2.3 The method of substitution . . . 13
2.4 Integration by parts . . . 13
2.5 Rational functions . . . 14
2.5.1 Constant divided by a linear function . . . 14
2.5.2 Constant divided by an irreducible quadratic function . . . 14
2.5.3 A linear function divided by an irreducible quadratic function . . . 14
2.5.4 Partial fraction decomposition . . . 14
2.5.5 All at once . . . 16
2.6 Riemann's integral . . . 17
2.7 Newton's integral . . . 19
2.8 Application – few words on probability . . . 20
2.8.1 Few words on discrete probability . . . 20
2.8.2 Continuous distribution function . . . 21
2.8.3 Exponential distribution . . . 21
2.8.4 Normal distribution . . . 22
2.9 Double Integrals . . . 22
2.10 Change of variables . . . 26
2.11 Polar coordinates . . . 29

1 Functions of two and more variables – extremes

Similarly to the one-dimensional case, we talk about local and global extremes.

Definition 1.1. *Let $f : M \subset \mathbb{R}^2 \to \mathbb{R}$. We say that f attains a* local maximum *at a point $(x_0, y_0) \in M^0$ if there is $r > 0$ such that $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in B_r(x_0, y_0)$.*
We say that f attains a local minimum *at a point $(x_0, y_0) \in M^0$ if there is $r > 0$ such that $f(x_0, y_0) \leq f(x, y)$ for all $(x, y) \in B_r(x_0, y_0)$.*

Definition 1.2. *Let $f : M \subset \mathbb{R}^2 \to \mathbb{R}$. We say that f attains its maximum on M at a point (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in M$. Similarly, f attains its minimum on M at a point (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all $(x, y) \in M$.*

1.1 Extremes

Assume $f \in C^1$ Let f has a local extrem at (x_0, y_0). Then $g(x) = f(x, y_0)$ has also a local extreme at x_0 and, therefore, $g'(x_0) = 0$. Similarly, $h(y) = f(x_0, y)$ has a local extreme at y_0 and thus $h'(y_0) = 0$. This leads to the following observation.

Observation 1.1. *Let $f \in C^1$ have a local extreme at (x_0, y_0). Then $\nabla f(x_0, y_0) = 0$.*

Definition 1.3. *A point $(x_0, y_0) \in M$ such that $\nabla f(x_0, y_0) = 0$ is called a* stationary point*.*

How to find all local extremes of given function?
Step 1: determine the stationary point.
Step 2: examine the possible extremes in the stationary point.
Reminder: in the one-dimensional case one has to treat the sign of the second derivative in order to decide if there is an extreme in a stationary point.

Example Let find all stationary points of $f(x, y) = x^2 - y^2$. We have $\nabla f(x, y) = (2x, -2y)$ and therefore the only stationary point is $(x_0, y_0) = (0, 0)$. Is there a maximum or minimum?

Observation 1.2. *Let $f \in C^2$ and let (x_0, y_0) be its stationary point. Then:*

1. *$\nabla^2 f$ positive-definite yields that (x_0, y_0) is a local minimum,*
2. *$\nabla^2 f$ negative-definite yields that (x_0, y_0) is a local maximum,*
3. *$\nabla^2 f$ indefinite means that (x_0, y_0) is a saddle point – there is no maximum nor minimum and*
4. *otherwise, we cannot decide.*

Example: Let go back to $f(x, y) = x^2 - y^2$. We already know that $(x_0, y_0) = (0, 0)$ is a stationary point. We have

$$\nabla^2 f = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus $\det \nabla^2 f(0, 0) = -4$ and there is no extreme at $(0, 0)$.

Another example Determine all local extremes of

$$f(x, y) = x^3 + 3xy^2 - 15x - 12y.$$

Step 1, stationary points:

$$\nabla f(x, y) = (3x^2 + 3y^2 - 15, 6xy - 12)$$

and we stationary points are solutions to

$$\begin{aligned} 3x^2 + 3y^2 - 15 &= 0 \\ 6xy - 12 &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} x^2 + y^2 - 5 &= 0 \\ xy &= 2. \end{aligned}$$

We deduce from the second equation that x and y are different from zero. The second equation yields $x = \frac{2}{y}$. We plug this into the first equation to deduce

$$\frac{4}{y^2} + y^2 - 5 = 0$$

which is equivalent to

$$y^4 - 5y^2 + 4 = 0.$$

We have $y^2 = 4$, $y^2 = 1$ and therefore there are four stationary points

$$A = (-1, -2),\ B = (1, 2),\ C = (2, 1),\ D = (-2, -1).$$

Step 2: We have

$$\nabla^2 f = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix}$$

Further,

$$\nabla^2 f(A) = \begin{pmatrix} -6 & -12 \\ -12 & -6 \end{pmatrix},\ \det \nabla^2 f(A) = -108$$

and A is a saddle point.

$$\nabla^2 f(B) = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix},\ \det \nabla^2 f(B) = -108$$

and B is a saddle point.

$$\nabla^2 f(C) = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix},\ \det \nabla^2 f(C) = 108$$

and C is a point of a local minimum. The value of the local minimum is $f(C) = -28$.

$$\nabla^2 f(D) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix},\ \det \nabla^2 f(D) = 108$$

and D is a point of a local maximum. The value of the local maximum is $f(D) = 28$.

1.2 Global extremes

Example A company manufactures two products A and B that sell for \$10 and \$9 per unit respectively. The cost of producing x units of A and y units of B is

$$400 + 2x + 3y + 0.01(3x^2 + xy + 3y^2).$$

How to find the values of x and y that maximize company's profit?
First of all, the company profit $P = P(x, y)$ is given as

$$P(x, y) = 10x + 9y - (400 + 2x + 3y + 0.01(3x^2 + xy + 3y^2)).$$

This in turn implies that

$$\nabla P(x, y) = (8 - 0.06x - 0.01y, 6 - 0.01x - 0.06y)$$

and the stationary point is

$$(x, y) = (120, 80).$$

Next,

$$\nabla^2 P(x, y) = \begin{pmatrix} -0.06 & -0.01 \\ -0.01 & -0.06 \end{pmatrix}$$

and one can use the Sylvester rule to deduce that this matrix is negative-definite. We deduced that $(120, 80)$ is the point of local maximum. Is it also a global one?

Definition 1.4. *A set $M \subset \mathbb{R}^n$ is* convex *if for every $x, y \in M$ and every $\lambda \in (0, 1)$ it holds that*

$$\lambda x + (1 - \lambda)y \in M.$$

Definition 1.5. *Let* Dom $f \subset \mathbb{R}^n$ *be a convex set. We say that f is convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in \mathrm{Dom}\, f$ and $\lambda \in (0, 1)$. The function is strictly convex, if

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

The function f is (strictly) concave if $-f$ is (strictly) convex.

Several observation

- The second gradient is positive (negative) definite – the function is strictly convex (concave).
- The function is convex on its domain – every local minimum is a global minimum.
- The function is concave on its domain – every local maximum is a global maximum.

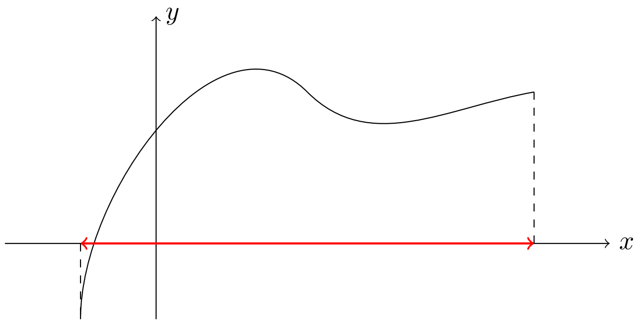
Now we go back to the example given in the beginning of this subsection. Since $\nabla^2 P$ is negative-definite, the function P (total profit) is concave and, in turn, the local maximum is simultanously also a global maximum.

1.3 Extrema with respect to a compact set

Definition 1.6. *A set $M \subset \mathbb{R}^2$ is* bounded *if there is $r > 0$ such that $M \subset B_r(0,0)$.*

Observation 1.3. *Let $f : M \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let M be a bounded and closed set. Then there is (x_0, y_0) where f attains its minimum on M and there is (x_1, y_1) where M attains its maximum.*

One-dimensional case, reminder
Consider this function: One has to consider separately the interior of M and the 'boundary' of



M. Although the function whose graph is in the picture has two stationary points, just one of them is a point of a global extreme. The point of the global minimum is on the edge of M.

Example
Lets find the extrema of

$$f(x,y) = (x^2 + y)e^y$$

on the set

$$M = \{(x,y) \in \mathbb{R}^2,\ y \geq \frac{1}{3}x,\ y \leq 3x,\ y \leq 5 - x\}.$$

We decompose the set into several parts and we treat each part separately:

- *Interior* $M^0 = \{(x,y) \in \mathbb{R}^2,\ y > \frac{1}{3}x,\ y < 3x,\ y < 5 - x\}$. There we compute

  $$\nabla f = (2xe^y,\ (x^2 + y + 1)e^y)$$

  and the only stationary point $(x,y) = (0,-1)$ is outside of the set M.

- *line number one* $M^1 = \{(x,y) \in \mathbb{R}^2,\ y = \frac{1}{3}x,\ y < 3x,\ y < 5 - x\}$. There we have

  $$g(x) := f(x, \frac{1}{3}x) = \left(x^2 + \frac{1}{3}x\right) e^{\frac{x}{3}}$$

  and $g'(x) = 0$ yields the stationary points with negative x and these does not belong to M.

- *line number two* $M^2 = \{(x,y) \in \mathbb{R}^2,\ y > \frac{1}{3}x,\ y = 3x,\ y < 5 - x\}$. Then

  $$g(x) := f(x, 3x) = \left(x^2 + 3x\right) e^{3x}$$

  and the stationary points are again outside of the set M.

- *line number three* $M^3 = \{(x,y) \in \mathbb{R}^2,\ y > \frac{1}{3},\ y < 3x,\ y = 5 - x\}$. Then
$$g(x) := f(x, 5-x) = (x^2 - x + 5)e^{5-x}$$
and there are no stationary points.
- The last part of the boundary which has to be taken into account consists of the vertices, i.e. the points
$$A = (0,0),\ B = \left(\frac{5}{4}, \frac{15}{4}\right),\ C = \left(\frac{15}{4}, \frac{5}{4}\right)$$

Eventually, the maximum is at B and its value is $\frac{85}{16}e^{15/4}$, the minimum is at A and its value is 0.

In case of more complicated boundary (something more difficult than just a line), we use the Lagrange multipliers method.

Theorem 1.1. *Let $f : \mathrm{Dom}\, f \subset \mathbb{R}^2 \to \mathbb{R}$ be of class C^1 and let it be defined on the neighborhood of a set M which is given as*
$$M = \{(x,y) \subset \mathbb{R}^2,\ g(x,y) = 0\}$$
for some function $g \in C^1$. Let $\nabla g \neq 0$. If there is an extreme of f with respect to the set M then there exists $\lambda \in \mathbb{R}$ such that
$$\nabla f + \lambda \nabla g = 0.$$

Example We show how to determine a maximum and minimum of $f(x,y) = -y^2 + x^2 + \frac{4}{3}x^3$ on the set $M = \{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 = 4\}$.
The first question is whether there is a maximum and minimum. Our first claim is that the set M is closed. Why? We define $g(x,y) = x^2 + y^2 - 4$ and then the set M is $g^{-1}(\{0\})$ and since $\{0\} \subset \mathbb{R}^2$ is a closed set, we deduce that M is also closed. Further, M is bounded since $M \subset B_3(0,0)$. Therefore, according to the very first observation of this talk there has to be a maximum and minimum of f on M.
Further, it holds that $\nabla g \neq 0$ for every $(x,y) \neq (0,0)$. Note that $(0,0) \notin M$ and thus we may use the Lagrange multipliers. We have
$$\nabla f(x,y) = (2x + 4x^2, -2y), \quad \nabla g(x,y) = (2x, 2y).$$
We end up with a system
$$\begin{aligned} 2x + 4x^2 + 2\lambda x &= 0 \\ -2y + 2\lambda y &= 0 \\ x^2 + y^2 &= 4. \end{aligned}$$
We deduce from the second equation that $y(2\lambda - 2) = 0$ and we get that either $y = 0$ or $\lambda = 1$. Consider first the case $y = 0$. Then the last equation yields $x^2 = 4$ and therefore $x = \pm 2$. We get two 'stationary' points
$$A = (2,0),\ B = (-2,0).$$
Next, assume $\lambda = 1$. The first equation then yields
$$4x + 4x^2 = 0$$

which gives $x = 0$ or $x = -1$.
Let $x = 0$. The last equation is then $y^2 = 4$ and we get $y = \pm 2$ and another two stationary points

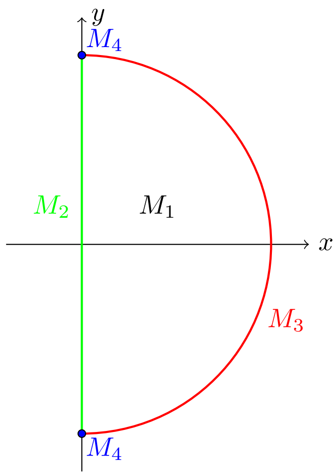
$$C = (0, 2),\ D = (0, -2).$$

Finally, let $x = -1$. Then we get $y^2 = 3$ and $y = \pm 3$ and we deduce another two stationary points

$$E = (-1, \sqrt{3}),\ D = (-1, -\sqrt{3}).$$

We have $f(A) = \frac{44}{3}$, $f(B) = -\frac{20}{3}$, $f(C) = -4$, $f(D) = -4$, $f(E) = -\frac{10}{3}$ and $f(F) = -\frac{10}{3}$. We deduce that the maximum is attained at the point $(2, 0)$ and its value is $\frac{44}{3}$, the minimum is attained at the point $(-2, 0)$ and its value is $-\frac{20}{3}$.

Example We find extremes of $f(x, y) = x^2 + y^2 - 12x + 16y$ on a set $M = \{(x, y) \subset \mathbb{R}^2,\ x^2 + y^2 \leq 25, x \geq 0\}$.



We dismantle the set into four pieces

$$\begin{aligned}
M_1 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25, x > 0\},\\
M_2 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25, x = 0\},\\
M_3 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 25, x > 0\},\\
M_4 &= \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 25, x = 0\}.
\end{aligned}$$

and we takcle each subset separately:
- Stationary points in $M_1 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25, x > 0\}$:
We solve $\nabla f = 0$ which is

$$\begin{aligned}
2x - 12 &= 0\\
2y + 16 &= 0
\end{aligned}$$

Therefore the stationary point is $(6, -8)$. However, $6^2 + (-8)^2 = 100 > 25$ and this point does not belong to M_1.
- Stationary points in $M_2 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 25, x = 0\}$:

We are going to consider a function $f(x,y)$ on line $x=0$. Therefore it is enough to examine function $f(0,y) =: h(y)$. We have

$$h(y) = y^2 + 16y$$

and therefore $h'(y) = 2y + 16$. The resulting stationary point is $x = 0$, $y = -8$. However, $(-8)^2 > 25$ and the point $(0,-8)$ does not belong to M_2.
- Stationary points in $M_3 = \{(x,y) \in \mathbb{R}^2, x^2+y^2 = 25, x = 0\}$:
There is a constraint $g(x,y) = x^2 + y^2 - 25$. We have $\nabla g = (2x, 2y)$ and we have $\nabla g \neq 0$ for every $(x,y) \in M_3$. The system $\nabla f + \lambda \nabla g = 0$ complemented with $g = 0$ has form

$$\begin{aligned} 2x - 12 + 2x\lambda &= 0 \\ 2y + 16 + 2y\lambda &= 0 \\ x^2 + y^2 &= 25. \end{aligned}$$

We may deduce that $y \neq 0$ (otherwise the second equation cannot be true) and $\lambda \neq 0$ (otherwise $x = 6$, $y = -8$ and the last equation is not fulfilled. The first and second equation might be rewritten as

$$\begin{aligned} x\lambda &= 6 - x \\ y\lambda &= -8 - y \end{aligned}$$

and we divide the first equation by the second to get

$$\frac{\lambda x}{\lambda y} = \frac{6-x}{-8-y}.$$

This yields

$$\frac{x}{y} = \frac{x-6}{8+y}$$

and

$$8x + xy = xy - 6y$$

and therefore

$$y = -\frac{4}{3}x.$$

We plug this into the last equation ($x^2 + y^2 = 25$) to get

$$x^2 + \frac{16}{9}x^2 = 25$$

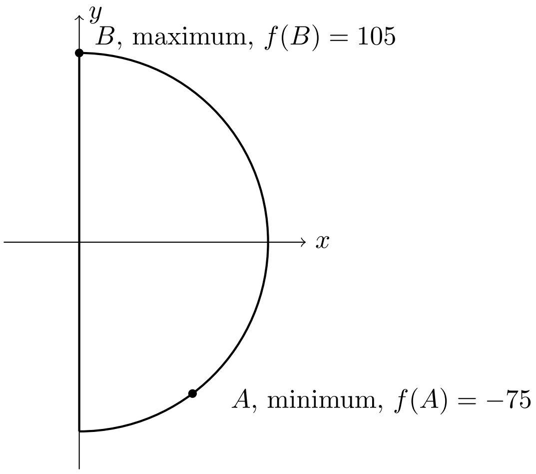
which yields $x = \pm 3$. Therefore we have two stationary points $(-3, 4)$ and $(3, -4)$, however, the first one does not belong to M_3. So we take into consideration only $A = (3, -4)$.
- Stationary points in $M_4 = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 = 25,\ x = 0\}$: This set consists only of two points. Indeed, let both equations holds at once. Then necessarily

$$y^2 = 25$$

and we have two points $B = (0, 5)$ and $C = (0, -5)$. These two points have to be considered as there might appear global extremes (although these points are not stationary).
- Final evaluation: We have just three points where the extremes might be attained:
$A = (3, -4)$, $B = (0, 5)$ and $C = (0, -5)$. We have

$$\begin{aligned} f(A) &= -75 \\ f(B) &= 105 \\ f(C) &= -55. \end{aligned}$$

We deduce that the minimum of f on set M is attained at the point $(3, -4)$ and its value is -75, the maximum of f on set M is attained at point $(0, 5)$ and its value is 105.



2 Integrals

2.1 Motivation, basic notions

Let us begin with few exercises:

- Recall that total revenue and marginal revenue are related as

$$MR = \frac{\partial TR}{\partial Q}.$$

  What is the total revenue function if

$$MR = 100 + 20Q + 3Q^2$$

  and $TR(2) = 260$?

  One can deduce (from the table of basic derivatives) that

$$TR = 100Q + 10Q^2 + Q^3 + C,$$

  where C might be an arbitrary constant. However, the condition $TR(2) = 2$ yields the precise value of C and thus the demanded total revenue is

$$TR = 100Q + 10Q^2 + Q^3 + 12.$$

- The train begins its motion with a velocity v (in meters per seconds) given as

$$v(t) = \frac{20t}{t+20}$$

  Determine the distance the train has traveled after 30 seconds.

  Everyone knows that the velocity is the derivative of the distance with respect to the time, i.e., $v(t) = s'(t)$. This particular integral is not that easy to find but you can verify that

$$s(t) = 20t - 400\log(t+20) + 400\log(20)$$

  is the correct solution. Indeed,

$$s'(t) = 20 - \frac{400}{t+20} = \frac{20t + 400 - 400}{t+20} = \frac{20t}{t+20}$$

  and also $s(0) = 0$. Thus, the answer is

$$s(30) = 600 - 400\log\left(\frac{50}{20}\right).$$

Definition 2.1. *We say that F is a* primitive function *(on interval (a,b)) of f if $F'(x) = f(x)$ (for all $x \in (a,b)$).*

We will use also the following notation

$$\int f(x)\ \mathrm{d}x = F(x).$$

Remark 2.1. *The primitive function is also called* indefinite integral *of* antiderivative *of f.*

Observation 2.1. *Let F_1 and F_2 be two primitive functions of f on interval $(a,b) \subset \mathbb{R}$. Then $F_1 - F_2 \equiv c$ for some constant $c \in \mathbb{R}$.*

Proof. It suffices to consider $(F_1 - F_2)' = (f - f) = 0$. The claim follows immediately. □

As a consequence the primitive function is not determined uniquely. In particular, the primitive function of a given function f is a whole set of functions which differ by an arbitrary constant – if F is the primitive function of f then all functions in the form $F + c$, $c \in \mathbb{R}$ are also primitive functions.

2.2 Calculation – basic methods

Observation 2.2. *Let F be the primitive function of f and G be the primitive function of g. Then $F + G$ is the primitive function of $f + g$ and cF is the primitive function of cf for every $c \in \mathbb{R}$.*

Proof. It is enough to use rules for the derivatives. □

Further, we may use the table of basic derivatives in an 'inverted' way:

$f(x)$	$F(x)$	conditions
x^n	$\frac{1}{n+1}x^{n+1} + c,\ c \in \mathbb{R}$	$n \neq -1$, x as usual
x^{-1}	$\log\lvert x\rvert + c,\ c \in \mathbb{R}$	$x \neq 0$
e^x	$e^x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
a^x	$\frac{1}{\log a}a^x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$, $a \in (0,1) \cup (1,\infty)$
$\sin x$	$-\cos x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
$\cos x$	$\sin x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
$\frac{1}{1+x^2}$	$\arctan x + c,\ c \in \mathbb{R}$	$x \in \mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c,\ c \in \mathbb{R}$	$x \in (-1,1)$

We present several exemplary calculations:

$$\int \frac{x+1}{\sqrt{x}}\ \mathrm{d}x = \int x^{\frac{1}{2}} + x^{-\frac{1}{2}}\ \mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c,\ \ c \in \mathbb{R},$$

$$\int \frac{x^2}{x^2+1}\ \mathrm{d}x = \int 1 - \frac{1}{1+x^2}\ \mathrm{d}x = x - \arctan x + c,\ \ c \in \mathbb{R},$$

$$\int \frac{2^{x+1} - 5^{x-1}}{10^x}\ \mathrm{d}x = \int 2\left(\frac{1}{5}\right)^x + \frac{1}{5}\left(\frac{1}{2}\right)^x\ \mathrm{d}x = \frac{2}{\log\frac{1}{5}}\left(\frac{1}{5}\right)^x + \frac{1}{5\log\frac{1}{2}}\left(\frac{1}{2}\right)^x + c,\ \ c \in \mathbb{R}.$$

The first example of somewhat more advanced methods is 'linear substitution':

Observation 2.3. *Let $F(x)$ be the primitive function of $f(x)$. Then $\frac{1}{a}F(ax+b)$ is the primitive function of $f(ax+b)$.*

Proof. Indeed, we derive the composed function $F(ax+b)$:

$$(F(ax+b))' = F'(ax+b)(ax+b)' = f(ax+b)a.$$

□

Below, we compute several exemplary exercises

$$\int (2x+3)^7 \, \mathrm{d}x = \frac{1}{16}(2x+7)^8 + c, \; c \in \mathbb{R},$$

$$\int \frac{1}{x^2+4} \, \mathrm{d}x = \int \frac{1}{4}\frac{1}{(x/2)^2+1} \, \mathrm{d}x = \frac{1}{2}\arctan(x/2) + c, \; c \in \mathbb{R}.$$

2.3 The method of substitution

Theorem 2.1. *Let $\varphi : (\alpha, \beta) \to (a, b)$ has a finite derivative in every $x \in (\alpha, \beta)$ and let f be defined on (a, b). Then*

$$\int f(x) \, \mathrm{d}x = \int f(\varphi(x))\varphi'(x) \, \mathrm{d}x.$$

Example: Solve $\int \sin^2 x \cos x \, \mathrm{d}x$:

$$\int \sin^2 x \cos x \, \mathrm{d}x = \begin{bmatrix} \sin x = t \\ \cos x \, \mathrm{d}x = 1 \, \mathrm{d}t \end{bmatrix} = \int t^2 \, \mathrm{d}t = \frac{t^3}{3} + c = \frac{\sin^3 x}{3} + c, \, c \in \mathbb{R}.$$

2.4 Integration by parts

Theorem 2.2. *Let F be the primitive function of f and G be the primitive function of g. Then*

$$\int F(x)g(x) \, \mathrm{d}x = F(x)G(x) - \int f(x)G(x) \, \mathrm{d}x$$

Exercises

- Integrals of the type $\int \textit{Polynomial}(e^x, \sin x, \cos x) \, \mathrm{d}x$:

$$\int xe^{3x} \, \mathrm{d}x = \begin{bmatrix} F(x) = x & g(x) = e^{3x} \\ f(x) = 1 & G(x) = \frac{1}{3}e^{3x} \end{bmatrix} = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x} \, \mathrm{d}x = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + c, \, c \in \mathbb{R}$$

- Integrals of the type $\int (e^x, \sin x, \cos x)(e^x, \sin x, \cos x) \, \mathrm{d}x$:

$$\begin{aligned} \int e^x \sin x \, \mathrm{d}x &= \begin{bmatrix} F(x) = e^x & g(x) = \sin x \\ f(x) = e^x & G(x) = -\cos x \end{bmatrix} = -e^x \cos x + \int e^x \cos x \, \mathrm{d}x \\ &= \begin{bmatrix} F(x) = e^x & g(x) = \cos x \\ f(x) = e^x & G(x) = \sin x \end{bmatrix} = -e^x \cos x + e^x \sin x - \int e^x \sin x \, \mathrm{d}x \end{aligned}$$

  and so we have just deduced that

$$\int e^x \sin x \, \mathrm{d}x = \frac{1}{2}\left(\sin x - \cos x\right) e^x + c, \; c \in \mathbb{R}.$$

- Integrals of the type $\int \textit{Polynomial}(\log x, \arctan x) \, \mathrm{d}x$:

$$\begin{aligned} \int x^3 \log x \, \mathrm{d}x = \begin{bmatrix} F(x) = \log x & g(x) = x^3 \\ f(x) = \frac{1}{x} & G(x) = \frac{1}{4}x^4 \end{bmatrix} &= \frac{1}{4}x^4 \log x - \frac{1}{4}\int x^4 \, \mathrm{d}x \\ &= \frac{1}{4}x^4 \log x - \frac{1}{20}x^5 + c, \, c \in \mathbb{R}. \end{aligned}$$

2.5 Rational functions

Rational functions are functions of the form

$$f(x) = \frac{R(x)}{P(x)}$$

where both $R(x)$ and $P(x)$ are polynomials. How to solve integrals of rational functions? Lets proceed in the algorithmic manner.

2.5.1 Constant divided by a linear function

The easiest case, it suffices to use the linear substitution. For example

$$\int \frac{5}{4-3x}\,\mathrm{d}x = -\frac{5}{3}\log(|4-3x|) + c,\ c \in \mathbb{R}.$$

2.5.2 Constant divided by an irreducible quadratic function

This is also solvable by the linear substitution, the result is certain arcus tangens this time. For example (note that the denominator does not have any real roots)

$$\int \frac{4}{x^2+2x+5}\,\mathrm{d}x = \int \frac{4}{(x+1)^2+4}\,\mathrm{d}x = \int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}\,\mathrm{d}x = 2\arctan\left(\frac{x+1}{2}\right) + c,\ c \in \mathbb{R}.$$

2.5.3 A linear function divided by an irreducible quadratic function

We start with the following observation whose proof is an easy consequence of the substitution method

Observation 2.4. *It holds that*

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \log|f(x)| + c,\ c \in \mathbb{R}.$$

We use this observation to transfer this case into the previous one. For example:

$$\begin{aligned}\int \frac{3x+1}{x^2+2x+2}\,\mathrm{d}x = \frac{3}{2}\int \frac{2x+\frac{2}{3}}{x^2+2x+2}\,\mathrm{d}x &= \frac{3}{2}\left(\int \frac{2x+2}{x^2+2x+2}\,\mathrm{d}x - \frac{4}{3}\int \frac{1}{x^2+2x+2}\right)\\ &= \frac{3}{2}\log(x^2+2x+2) - \frac{4}{3}\int \frac{1}{(x+1)^2+1}\,\mathrm{d}x\\ &= \frac{3}{2}\log(x^2+2x+2) - \frac{4}{3}\arctan(x+1) + c,\ c \in \mathbb{R}.\end{aligned}$$

2.5.4 Partial fraction decomposition

We assume here that the polynomial in the numerator is of lower order than the polynomial in the denominator. The partial fraction decomposition starts with the following theorem.

Theorem 2.3. *Every polynomial can be written as a product of* $1-$*degree polynomials and irreducible* $2-$*degree polynomials.*

Recall that a polynomial $ax^2 + bx + c$ is irreducible if there are no real roots.
We adopt the following strategy: the polynomial Q in the denominator may be written as a product of the aforementioned polynomials. In that case, the whole fraction is rewritten as a sum of fractions with $1-$ and $2-$ degree polynomials in the denominator (partial fraction decomposition). This sum may be integrated by methods mentioned in the previous talks.

Below we show how to deal with $1-$degree polynomials. Let compute $\int \frac{x+1}{x^2+5x+6}\, \mathrm{d}x$. We know that $(x^2 + 5x + 6) = (x + 2)(x + 3)$ and thus

$$\begin{aligned} \frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)} &= \frac{A}{x+2} + \frac{B}{x+3} \\ &= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}. \end{aligned}$$

This yields
$$x + 1 = Ax + 3A + Bx + 2B$$
and we compare appropriate coefficients to deduce

$$\begin{aligned} 1 &= A + B \\ 1 &= 3A + 2B \end{aligned}$$

Thus $A = -1$ and $B = 2$ and, consequently,

$$\frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)} = \frac{2}{x+3} - \frac{1}{x+2}.$$

Thus

$$\int \frac{x+1}{x^2+5x+6}\, \mathrm{d}x = 2\int \frac{1}{x+3}\, \mathrm{d}x - \int \frac{1}{x+2}\, \mathrm{d}x = 2\log|x+3| - \log|x+2| + c,\ c \in \mathbb{R}$$

What happens if there is a one-degree polynomial powered to some number higher than 1? This is shown in the following example. Let compute $\int \frac{3x^2-2x}{(x-1)^2(2x-1)}\, \mathrm{d}x$. This time we write

$$\begin{aligned} \frac{3x^2-2x}{(x-1)^2(2x-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x-1} \\ = \frac{A(x-1)(2x-1) + B(2x-1) + C(x-1)^2}{(x-1)^2(2x-1)}. \end{aligned}$$

We deduce
$$3x^2 - 2x = A(2x^2 - 3x + 1) + B(2x - 1) + C(x^2 - 2x + 1)$$
and thus

$$\begin{aligned} 3 &= 2A + C \\ -2 &= -3A + 2B - 2C \\ 0 &= A - B + C \end{aligned}$$

Let compute $\int \frac{6x+4}{(x^2+2x+2)(x-1)}\,\mathrm{d}x$. This time we have

$$\frac{6x+4}{(x^2+2x+2)(x-1)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+C(x^2+2x+2)}{(x^2+2x+2)(x-1)}$$

and we deduce

$$\begin{aligned} 0 &= A + C \\ 6 &= -A + B + 2C \\ 4 &= -B + 2C \end{aligned}$$

We deduce $A = -2$, $B = 0$ and $C = 2$ and thus

$$\int \frac{6x+4}{(x^2+2x+2)(x-1)}\,\mathrm{d}x = 2\int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{2x}{x^2+2x+2}\,\mathrm{d}x$$

The first integral is simple so we pay attention to the second one. We have

$$\begin{aligned} \int \frac{2x}{x^2+2x+2}\,\mathrm{d}x = \int \frac{2x+2}{x^2+2x+2}\,\mathrm{d}x - 2\int \frac{1}{(x+1)^2+1}\,\mathrm{d}x \\ = \log(x^2+2x+2) - 2\arctan(x+1) + c,\ \ c \in \mathbb{R} \end{aligned}$$

Thus

$$\int \frac{6x+4}{(x^2+2x+2)(x-1)}\,\mathrm{d}x = 2\log|x-1| - \log(x^2+2x+2) + 2\arctan(x+1) + c,\ \ c \in \mathbb{R}.$$

The above attitude (partial fraction decomposition) may be summarized as follows

Theorem 2.4. *Let* $\deg P < \deg Q$ *and let*

$$Q(x) = \alpha_0(x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k}(x^2+p_1x+q_1)^{s_1} \cdot \ldots \cdot (x^2+p_lx+q_l)^{s_l}$$

where the second order polynomials have no real roots and no multiplier divide any other one and all coefficients are integers. Then there are real numbers $A_{11}, \ldots, A_{1r_1}, \ldots, A_{k1}, \ldots, A_{kr_k}$ *and* $B_{11}, C_{11}, \ldots, B_{1s_1}, C_{1s_1}, \ldots B_{l1}, C_{l1}, \ldots, B_{ls_l}, C_{ls_l}$ *such that*

$$\begin{aligned} \frac{P(x)}{Q(x)} = \frac{A_{11}}{x-\alpha_1} + \ldots + \frac{A_{1r_1}}{(x-\alpha_1)^{r_1}} + \ldots + \frac{A_{k1}}{(x-\alpha_k)} \\ + \ldots + \frac{A_{kr_k}}{(x-\alpha_k)^{r_k}} + \frac{B_{11}x+C_{11}}{x^2+p_1x+q_1} + \ldots + \frac{B_{1s_1}x+C_{1s_1}}{(x^2+p_1x+q)^{s_1}} \\ + \ldots + \frac{B_{l1}x+C_{l1}}{x^2+p_lx+q} + \ldots + \frac{B_{ls_l}x+C_{ls_l}}{(x^2+p_lx+q_l)^{s_l}} \end{aligned}$$

This theorem is presented without a proof.

2.5.5 All at once

In order to deal with the most general case, one has reduce the degree of the polynomial in the numerator and then it is possible to use the above methods. Let compute the following

$$\int \frac{x^3}{x^2-1}\,\mathrm{d}x.$$

One deduce (by division or proper guess) that

$$\frac{x^3}{x^2-1} = x - \frac{x}{(x-1)(x+1)}$$

and we utilize the partial fraction decomposition in order to get

$$\frac{x^3}{x^2-1} = x - \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}.$$

Consequently,

$$\int \frac{x^3}{x^2-1}\,\mathrm{d}x = \frac{x^2}{2} - \frac{1}{2}\log(|x-1|) - \frac{1}{2}\log(|x+1|) + c,\ c \in \mathbb{R}.$$

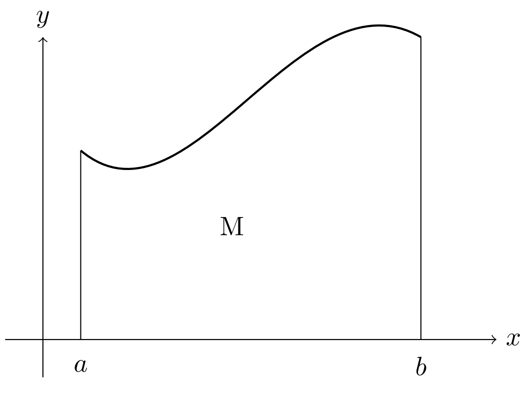
2.6 Riemann's integral

The main aim of this section is to compute the area which is bounded by a graph of function. More precisely, let f be a positive function defined on an interval (a, b). We will try to compute the area of a set

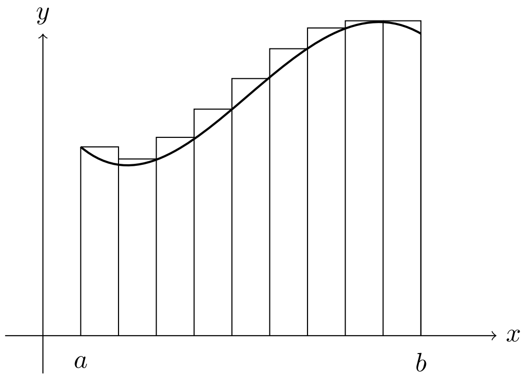
$$M = \{\langle x, y\rangle \in \mathbb{R}^2,\ x \in (a, b),\ 0 < y < f(x)\}. \tag{1}$$

The area is easy assuming $f \equiv c$, $c > 0$. In that case the area is given by $c(b - a)$.

In what follow, we show how to compute an area of the following set:

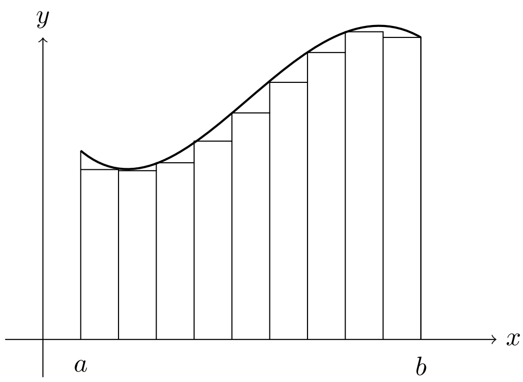


What if f is non-constant? We can approximate the value of the area by several rectangles as you can see on the following picture



Clearly, the area of M is less than the constructed approximation, however once there will be enough small rectangles, the approximation will be close to the true value.

We can also try to use the following approximation – this time we use maximal rectangle which are inside of the set M



In this case we obtain an area which is less than the area of M.

This idea is summarized in the following definition.

Definition 2.2. *Let f be a real function defined on $[a, b]$. We define sequences*

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{b-a}{n} \min\{f(x), x \in [a + (i-1)(b-a)/n, a + i(b-a)/n]\} \\ S_n &= \sum_{i=1}^{n} \frac{b-a}{n} \max\{f(x), x \in [a + (i-1)(b-a)/n, a + i(b-a)/n]\} \end{aligned} \tag{2}$$

If $\lim s_n = \lim S_n =: s$ then we say that s is the Riemann integral *of f over (a, b). We write*

$$(\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x = s$$

Let compute $\int_1^2 x^2\ \mathrm{d}x$: First, we divide $[1, 2]$ to n subintervals with length $\frac{1}{n}$. Namely, the $i-$th subinterval is of the form

$$\left[1 + \frac{i-1}{n}, 1 + \frac{i}{n}\right].$$

Clearly, the maximum value of x^2 on this interval is $(1+\frac{i}{n})^2$, the minimum value is $(1+\frac{i-1}{n})^2$. We get

$$\begin{aligned} s_n =& \sum_{i=1}^{n} \frac{1}{n}(1+\frac{i-1}{n})^2 \\ S_n =& \sum_{i=1}^{n} \frac{1}{n}(1+\frac{i}{n})^2 \end{aligned}$$

and we use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to obtain

$$\begin{aligned} s_n =& 1+\frac{n-1}{n}+\frac{(n-1)(2n-1)}{6n^2} \\ S_n =& 1+\frac{n+1}{n}+\frac{(n+1)(2n+1)}{6n^2} \end{aligned}$$

Since $\lim s_n = \lim S_n = \frac{7}{3}$ we deduce that this is the demanded area of the given set.

2.7 Newton's integral

Definition 2.3. *Let F be an antiderivative of f. Then*

$$(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x = F(b) - F(a).$$

The number $\int_a^b f(x)\ \mathrm{d}x$ is called the Newton integral of f over (a,b).

We use the notation $[F(x)]_a^b$ for the difference $F(b) - F(a)$.
The following theorem is presented without proof.

Theorem 2.5 (The basic theorem of calculus)**.** *Let f be defined on $[a,b]$ and let $(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x$ and $(\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x$ exist. Then*

$$(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x = (\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x$$

Let us note several remarks:

- This provides a simple way how to compute an area of the set M defined in (1).
- As the Riemann and Newton integrals are equal we write simply $\int_a^b f(x)\ \mathrm{d}x$ instead of $(\mathcal{R}) - \int_a^b f(x)\ \mathrm{d}x$ or $(\mathcal{N}) - \int_a^b f(x)\ \mathrm{d}x$.

Definition 2.4. *Let f defined on (a,b) have antiderivative F. The number*

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{x\to a-} F(x) - \lim_{x\to b+} F(x)$$

is called thegeneralized Newton integral of f over (a,b).

Once again, we will use $[F(x)]_a^b$ for $\lim_{x\to a-} F(x) - \lim_{x\to b+} F(x)$.

With this at hand it is easy to compute areas of certain sets. Consider the integral from the previous section:

$$\int_1^2 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}.$$

Substitution method for the Newton integral: One can also change variables when computing the Newton integral, nevertheless, it is necessary to compute also new bounds. Let compute several exercises:

Exercises

- Compute
$$\int_0^{\pi/2} \cos x \sin^3 x \, \mathrm{d}x.$$
Clearly, we use the change of variable $t = \sin x$ as then $\mathrm{d}t = \cos x \mathrm{d}x$. Note also that $t = 0$ for $x = 0$ and $t = 1$ for $x = \pi/2$. So
$$\int_0^{\pi/2} \cos x \sin^3 x \, \mathrm{d}x = \int_0^1 t^3 \, \mathrm{d}t = \frac{1}{4} \left[t^4\right]_0^1 = \frac{1}{4}.$$

- Evaluate
$$\int_1^4 \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x.$$
First of all, the integrand is not well defined for $x \in (1, 4)$ and, therefore, there is nothing to compute. This integral is wrongly given.

- Evaluate
$$\int_0^1 \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x.$$
Now it is correctly given, the integrand has sense for $x \in (0, 1)$. We change the variable as $t = 1 - x^2$ and therefore $\mathrm{d}t = -2x\mathrm{d}x$ and $t = 1$ for $x = 0$ and $t = 0$ for $x = 1$. Therefore
$$\int_0^1 \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x = -\frac{1}{2} \int_1^0 \frac{1}{\sqrt{t}} \, \mathrm{d}t = \frac{1}{2} \int_0^1 t^{-1/2} \, \mathrm{d}t = \left[t^{1/2}\right]_0^1 = 1.$$
where the additional minus appeared because we changed the bounds.

2.8 Application – few words on probability

In what follows, X is used mainly to denote a random variable. Since this is not a probability lesson, we do not effort to be rigorous and we admit certain level of intuition.

2.8.1 Few words on discrete probability

Definition 2.5. *Let X be a real-valued random variable. A* cumulative distribution function *is a function $F(x)$ given as $F(x) = P(X \leq x)$.*

Exercises

- Let there be usual 6 sided fair die. Draw a graph of the cumulative distribution function. What is the probability that the result is lower or equal to 2? What is the probability of an odd result? What is the expectation?

- The odds for the tennis match between Daniil Medvedev and Carlos Alcaraz are 3.23 for Medvedev and 1.38 for Alcaraz. Assuming the probability that Medvedev wins is 30 percent, if we have \$100 and we put \$40 on Medvedev and \$60 on Alcaraz, what is the expected profit? Is there any other distribution of bets which will maximize the profit?

2.8.2 Continuous distribution function

Exercise

- Two friends want to meet under the tail between 1PM and 2PM. They do not specify the exact time but if one of them come there, he will wait 10 minutes for the second one. What is the probability that they will meet each other?

Definition 2.6. *Let X be a real valued random variable. The function $f(x)$ such that $P(X \leq x) = \int_{-\infty}^{x} f(s)\,\mathrm{d}s$ is called a* probability density*. Clearly, $f(x) \geq 0$ for every $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$.*
The expectation *of a function $g(x)$ is $\int_{-\infty}^{\infty} g(s)f(s)\,\mathrm{d}s =: E(g(X))$.*

Exercise

- The tram line number 8 departs from the stop every 10 minutes in the morning. Calculate the probability that you will wait for it for more than 7 minutes in the morning. What is the expected time of waiting?

2.8.3 Exponential distribution

The distribution whose density is given as

$$f(x) = \begin{cases} 0 \text{ for } x < 0 \\ \lambda e^{-\lambda x} \text{ for } x \geq 0. \end{cases}$$

Is this a probability density? And what is its expectation?

Clearly, $f(x)$ is non-negative function and thus one has to check whether its integral over the real line is one.

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x = \left[-e^{-\lambda x}\right]_0^{\infty} = 0 + 1 = 1$$

and we have just deduced that f is indeed a probability density. To compute the expectation, we use the integration by parts in the following way:

$$\int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x = \lambda \int_{0}^{\infty} xe^{-\lambda x}\,\mathrm{d}x = \left[\begin{array}{cc} F(x) = x & g(x) = e^{-\lambda x} \\ f(x) = 1 & G(x) = -\frac{1}{\lambda}e^{-\lambda x} \end{array}\right]$$
$$= -\left[xe^{-\lambda x}\right]_0^{\infty} + \int e^{-\lambda x}\,\mathrm{d}x = \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty} = \frac{1}{\lambda}.$$

Exercise

- The bulb manufacturer Edison knows that the average lifespan of Edison bulbs is 10,000 hours. As part of its promotional campaign, it wants to guarantee a time T during which no more than 3 percents of the bulbs burn out. Determine this time. (Use the exponential distribution to model the lifespan of bulbs.)

We use the exponential distribution with $\lambda = \frac{1}{10{,}000}$. Recall that

$$P(X \leq x) = \int_0^x \frac{1}{10,000} e^{-\frac{1}{10{,}000}s}\,\mathrm{d}s$$

and we are looking for such x that

$$\int_0^x \frac{1}{10,000} e^{-\frac{1}{10{,}000}s}\,\mathrm{d}s \leq 0.03.$$

2.8.4 Normal distribution

The distribution whose density is given as

$$u(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}.$$

What is the expectation of the random function given by this distribution?
Generalization:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Here μ is the expectation and σ is the standard deviation of the random variable.
Exercises

- Let U be a real random variable whose density is given by u. Determine the following probabilities:

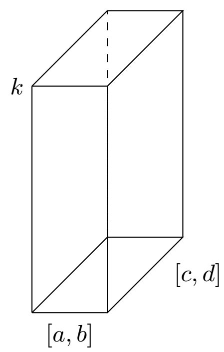
$$P(U < 1.67),\ P(U > 0.35),\ P(-1.5 < U < 0.5).$$

- Find x such that

$$a)\, P(U < x) = 0.9,\ b)\, P(U > x) = 0.8,\ c)\, P(-x < U < x) = 0.9.$$

2.9 Double Integrals

Let start with a motivation - double integral over a rectangle: Assume we have a constant function $f(x, y) \equiv k > 0$ on a set $M = [a, b] \times [c, d]$. What is the volume of a prism $[a, b] \times [c, d] \times [0, k]$? Simple answer is $(b - a)(c - d)k$. In this particular case we write $\int_{[a,b]\times[c,d]} f(x, y)\ \mathrm{d}x\mathrm{d}y =$



$(b - a)(c - d)k$.

Let M be a rectangle $[a, b] \times [c, d]$ and let $f(x, y)$ be a positive function defined on M. The value of the integral

$$\int_M f(x, y)\ \mathrm{d}x\mathrm{d}y$$

is a volume of a set

$$S = \{(x, y, z) \in \mathbb{R}^3,\ (x, y) \in M,\ 0 \leq z \leq f(x, y)\}.$$

Observation Let f be continuous function on a rectangle M. Then there is an integral

$$\int_M f(x, y)\ \mathrm{d}x\mathrm{d}y.$$

Remark 2.2 (measurable sets)**.** *It is not necessary to define integrals only over rectangles. In particular, the set M can be 'almost arbitrary' and then the meaning of integral is the same as in the previous slide. The only condition is that the integral*

$$\int_M 1\ \mathrm{d}x\mathrm{d}y$$

has value (and it might be even infinity). Such sets are called measurable sets *and we will not define them in the scope of this class. Let me just mention that not every set is measurable. On the other hand, it is very difficult to construct a non-measurable set. All sets appearing in this class are measurable. Interested students might look for the Banach-Tarski theorem.*

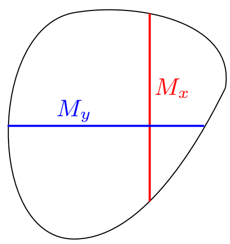
Remark 2.3 (measurable functions)**.** *Similarly, it is not necessary to define integrals only for continuous functions. Once again, there are functions called 'measurable functions' (and all continuous functions are measurable as well). And, similarly as before, it is very difficult to construct a non-measurable functions. In particular, every 'well-behaved' function is a measurable function and all functions appearing in this class are measurable.*

Definition 2.7. *Let $M \subset \mathbb{R}^2$. We define a* vertical cross-section *as*

$$M_x = \{y \in \mathbb{R},\ (x, y) \in M\}.$$

Similarly, we define a horizontal cross-section *as*

$$M_y = \{x \in \mathbb{R},\ (x, y) \in M\}.$$



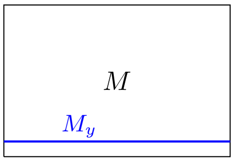
Theorem 2.6 (Fubini)**.** *Let $M \subset \mathbb{R}^2$ is a measurable set and $f : M \to \mathbb{R}$ be a measurable function. Then*

$$\int_M f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{M_x} f(x,y)\ \mathrm{d}y \right)\ \mathrm{d}x = \int_{\mathbb{R}} \left(\int_{M_y} f(x,y)\ \mathrm{d}x \right)\ \mathrm{d}y.$$

assuming that the integral on the left hand side is well defined.

Example

- Compute
$$\int_M 5x^2y - 2y^3\ \mathrm{d}x\mathrm{d}y, \quad M = [2,5] \times [1,3].$$
We use notation $f(x,y) = 5x^2y - 2y^3$ and we have $M_y = [2,5]$ for $y \in [1,3]$ and $M_y = \emptyset$



otherwise. Thus
$$\int_M f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \int_{M_y} f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_1^3 \int_2^5 f(x,y)\ \mathrm{d}x\mathrm{d}y.$$
We have
$$\begin{aligned}\int_M 5x^2y - 2y^3\ \mathrm{d}x\mathrm{d}y &= \int_1^3 \left(\int_2^5 5x^2y - 2y^3\ \mathrm{d}x \right)\ \mathrm{d}y \\ &= \int_1^3 \left[\frac{5x^3y}{3} - 2xy^3 \right]_{x=2}^{x=5}\ \mathrm{d}y = \int_1^3 \frac{625}{3}y - 10y^3 - \frac{40}{3}y + 4y^3\ \mathrm{d}y \\ &= \int_1^3 195y - 6y^3\ \mathrm{d}y = \left[\frac{195}{2}y^2 - \frac{3}{2}y^4 \right]_1^3 = 660.\end{aligned}$$

- Let compute integral
$$\int_M 2xe^y\ \mathrm{d}x\mathrm{d}y, \quad M = [0,2] \times [0,1].$$
We use the Fubini theorem to deduce
$$\int_M 2xe^y\ \mathrm{d}x\mathrm{d}y = \int_0^2 \left(\int_0^1 2xe^y\ \mathrm{d}y \right)\ \mathrm{d}x =$$
and since $2x$ is not a function of y, it can be put in front of the inner integral to obtain
$$= \int_0^2 2x \left(\int_0^1 e^y\ \mathrm{d}y \right)\ \mathrm{d}x = \int_0^2 2x\ \mathrm{d}x \int_0^1 e^y\ \mathrm{d}y.$$

Observation 2.5. *Let $f(x,y) = g(x)h(y)$ and let $M = [a,b] \times [c,d]$. Then*

$$\int_M f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_a^b g(x)\ \mathrm{d}x \int_c^d h(y)\ \mathrm{d}y.$$

Back to the given integral. We have

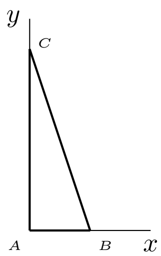
$$\int_M 2xe^y\ \mathrm{d}x\mathrm{d}y = \int_0^2 2x\ \mathrm{d}x \int_0^1 e^y\ \mathrm{d}y = \left[x^2\right]_0^2 \left[e^y\right]_0^1 = 4(e-1)$$

Example

- Compute

$$\int_M x^2 + xy - 1\ \mathrm{d}x\mathrm{d}y$$

where M is a triangle with vertices $A = \langle 0,0\rangle$, $B = \langle 2,0\rangle$ and $C = \langle 0,6\rangle$. Recall that



the line BC has an equation $y = 6 - 3x$. Therefore, the vertical cross-section has form $M_x = (0, 6-3x)$. and we deduce

$$\int_M x^2 + xy - 1\ \mathrm{d}x\mathrm{d}y = \int_0^2 \left(\int_0^{6-3x} x^2 + xy - 1\ \mathrm{d}y\right)\ \mathrm{d}x$$

$$= \int_0^2 \left[x^2 y + \frac{xy^2}{2} - y\right]_{y=0}^{y=6-3x}\ \mathrm{d}x$$

$$= \int_0^2 6x^2 - 3x^3 + 18x - 18x^2 + \frac{9}{2}x^3 - 6 + 3x\ \mathrm{d}x$$

$$= \left[\frac{3}{8}x^4 - 4x^3 + \frac{21}{2}x^2 - 6x\right]_0^2 = 6 - 32 + 42 - 12 = 4.$$

Observation 2.6 (Properties of integral)**.** *The following holds:*

- *Let f and g be (measurable) functions of two variables, $M \subset \mathbb{R}^2$ measurable set and $\alpha \in \mathbb{R}$. Then*

$$\int_M \alpha f(x,y) + g(x,y)\ \mathrm{d}x\mathrm{d}y = \alpha \int_M f(x,y)\ \mathrm{d}x\mathrm{d}y + \int_M g(x,y)\ \mathrm{d}x\mathrm{d}y.$$

- *Let $M = \bigcup_{i=1}^n M_i$. Then*

$$\int_M f(x,y)\ \mathrm{d}x\mathrm{d}y = \sum_{i=1}^n \int_{M_i} f(x,y)\ \mathrm{d}x\mathrm{d}y.$$

- *Let f be a measurable function, $M \subset \mathbb{R}^2$ be a measurable set and let f be non-negative on M (i.e. $f(x,y) \geq 0$ for all $(x,y) \in M$). Then*

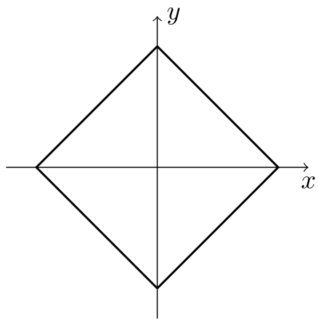
$$\int_M f(x,y)\ \mathrm{d}x\mathrm{d}y \geq 0.$$

Example

- Compute

$$\int_M (x+y)^2\ \mathrm{d}x\mathrm{d}y$$

where M is a square with vertices $A = \langle 0,-1\rangle$, $B = \langle 1,0\rangle$, $C = \langle 0,1\rangle$, $D = \langle -1,0\rangle$. Here



we divide M into two subsets, $M_1 = M \cap \{x < 0\}$ and $M_2 = M \cap \{x \geq 0\}$. We have

$$(M_1)_x = (-x-1, x+1) \text{ for } x \in (-1,0),$$
$$(M_2)_x = (x-1, -x+1) \text{ for } x \in [0,1).$$

Therefore,

$$\int_M (x+y)^2\ \mathrm{d}x\mathrm{d}y = \int_{-1}^{0} \left(\int_{-x-1}^{x+1} x^2 + 2xy + y^2\ \mathrm{d}y \right)\ \mathrm{d}x + \int_0^1 \left(\int_{x-1}^{-x+1} x^2 + 2xy + y^2\ \mathrm{d}y \right)\ \mathrm{d}x.$$

We compute

$$\int_{-1}^{0} \left(\int_{-x-1}^{x+1} x^2 + 2xy + y^2\ \mathrm{d}y \right)\ \mathrm{d}x = \int_{-1}^{0} \left[x^2y + xy^2 + \frac{y^3}{3} \right]_{y=-x-1}^{y=x+1}\ \mathrm{d}x$$
$$= \int_{-1}^{0} \frac{8}{3}x^3 + 4x^2 + 2x + \frac{2}{3}\ \mathrm{d}x = \left[\frac{2}{3}x^4 + \frac{4}{3}x^3 + x^2 + \frac{2}{3}x \right]_{-1}^{0} = \frac{1}{3}.$$

Similarly

$$\int_0^1 \left(\int_{x-1}^{1-x} x^2 + 2xy + y^2\ \mathrm{d}y \right)\ \mathrm{d}x = \int_0^1 \left[x^2y + xy^2 + \frac{y^3}{3} \right]_{y=x-1}^{y=1-x}\ \mathrm{d}x$$
$$= \int_0^1 -\frac{8}{3}x^3 + 4x^2 - 2x + \frac{2}{3}\ \mathrm{d}x = \left[-\frac{2}{3}x^4 + \frac{4}{3}x^3 - x^2 + \frac{2}{3}x \right]_0^1 = \frac{1}{3}.$$

Eventually, we obtain

$$\int_M (x+y)^2 \ \mathrm{d}x\mathrm{d}y = \frac{2}{3}.$$

2.10 Change of variables

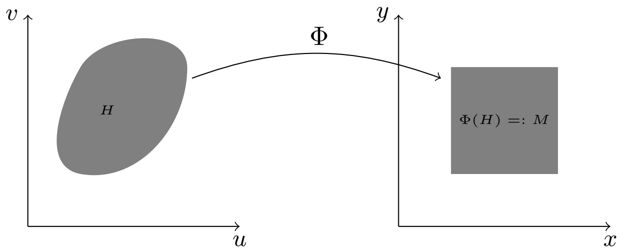
Recall one of the previous exercises:

$$\int_M (x+y)^2 \ \mathrm{d}x\mathrm{d}y, M \text{ is a square with vertices } A=(0,-1),$$
$$B=(1,0),\ C=(0,1),\ D=(-1,0).$$

The value of the integral is $\frac{2}{3}$ – that was deduced in the previous section, however, the computation was quite cumbersome. This time we present one better method how to compute the integral.
Recall, that the one-dimensional substitution method works in the following way

$$\int_a^b f(t) \ \mathrm{d}t = \int_\alpha^\beta f(\varphi(x))\varphi'(x) \ \mathrm{d}x.$$

This time, we consider a mapping $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(u,v) = (\varphi(u,v), \psi(u,v))$ and we assume that



$x = \varphi(u,v)$, $y = \psi(u,v)$.

Definition 2.8. *A mapping $\Phi: H \subset \mathbb{R}^2 \mapsto \mathbb{R}^2$ satisfying*

- $\Phi \in C^1$,
- Φ *is an injection,*
- *The Jacobian matrix of* Φ *is regular,*

is called a regular mapping.

Definition 2.9. *Let $\Phi: H \subset \mathbb{R}^2 \mapsto \mathbb{R}^2$ have components $\varphi(u,v)$ and $\psi(u,v)$. Then the* Jacobian matrix *of Φ is a matrix*

$$J\Phi(u,v) = \begin{pmatrix} \frac{\partial\varphi}{\partial u}(u,v) & \frac{\partial\varphi}{\partial v}(u,v) \\ \frac{\partial\psi}{\partial u}(u,v) & \frac{\partial\psi}{\partial v}(u,v) \end{pmatrix}.$$

Its determinant is then called the Jacobian determinant.

Theorem 2.7. *Let $f(x,y)$ is a measurable function on $D \subset \mathbb{R}^2$ and let $\Phi = (\varphi, \psi) : H \subset \mathbb{R}^2 \mapsto M$ is a regular mapping. Then*

$$\int_M f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_H f(\varphi(u,v), \psi(u,v))|\det J\Phi(u,v)|\ \mathrm{d}u\mathrm{d}v.$$

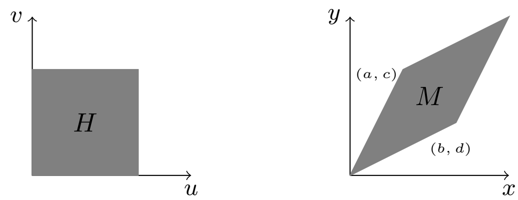
To show the role of the Jacobian determinant we consider a mapping

$$\begin{aligned} x &= au + bv =: \varphi(u,v) \\ y &= cu + dv =: \psi(u,v). \end{aligned}$$

Here we have that

$$J\Phi(u,v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let $H = (0,1) \times (0,1)$. Then M is a parallelogram with sides represented by vectors (a,c) and (b,d).



The area of H is $\int_H 1\ \mathrm{d}u\mathrm{d}v = 1$ and the area of M is $\int_M 1\ \mathrm{d}x\mathrm{d}y = \left|\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|$. Indeed, the area of the parallelogram is equal to $S = \sin\alpha \|(a,c)\| \|(b,d)\|$ and we may compute

$$\begin{aligned} S^2 = \sin^2\alpha \|(a,c)\|^2 \|(b,d)\|^2 &= \left(1 - \cos^2\alpha\right) \|(a,c)\|^2 \|(b,d)\|^2 \\ &= \|(a,c)\|^2 \|(b,d)\|^2 - ((a,c)\cdot(b,d))^2 \\ &= (a^2+c^2)(b^2+d^2) - (ab+cd)^2 = a^2d^2 + c^2b^2 - 2abcd \\ &= (ad-bc)^2. \end{aligned}$$

Therefore, there has to be a factor $|\det J\Phi|$ in order to get

$$\int_M 1\ \mathrm{d}x\mathrm{d}y = \int_H 1|\det J\Phi|\ \mathrm{d}u\mathrm{d}v.$$

Example:

- Let compute (once again) the integral

$$\int_M (x+y)^2\ \mathrm{d}x\mathrm{d}y$$

where $M = \{(x,y) \in \mathbb{R}^2,\ -1 \le x+y \le 1,\ -1 \le x-y \le 1\}$. We establish new variables

$$\begin{aligned} u &= x + y \\ v &= x - y \end{aligned}$$

and we deduce that

$$x = \frac{1}{2}(u+v)$$
$$y = \frac{1}{2}(u-v).$$

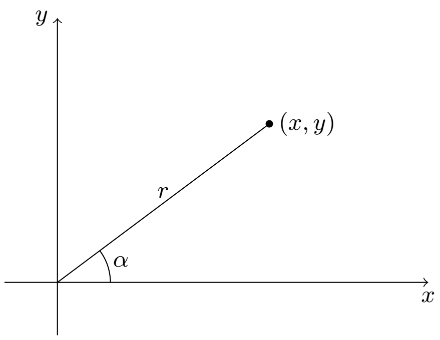
Thus, in this case, $\Phi(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$ and it holds that

$$J\Phi(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \ |\det J\Phi(u,v)| = \frac{1}{2}.$$

Let also mention that $\Phi(M) = \{(u,v) \in \mathbb{R}^2, \ u \in [-1,1], \ v \in [-1,1]\}$. Therefore we deduce that

$$\int_M (x+y)^2 \ \mathrm{d}x\mathrm{d}y = \int_{[-1,1]^2} u^2 \frac{1}{2} \ \mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_{-1}^{1} u^2 \ \mathrm{d}u \int_{-1}^{1} 1 \ \mathrm{d}v = \int_{-1}^{1} u^2 \ \mathrm{d}u = \left[\frac{u^3}{3}\right]_{-1}^{1} = \frac{2}{3}.$$

2.11 Polar coordinates



We have

$$x = r\cos\alpha$$
$$y = r\sin\alpha.$$

(and also $r = \sqrt{x^2+y^2}$). Therefore we establish $\Phi(r,\alpha) = (r\cos\alpha, r\sin\alpha)$ and we infer

$$J\Phi(r,\alpha) = \begin{bmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{bmatrix}$$

and

$$\det J\Phi(r,\alpha) = r\cos^2\alpha + r\sin^2\alpha = r.$$

Thus,

$$\int_M f(x,y) \ \mathrm{d}x\mathrm{d}y = \int_{\Phi^{-1}(M)} f(r\cos\alpha, r\sin\alpha) r \ \mathrm{d}r\mathrm{d}\alpha.$$

Examples:

- Volume of the ball with radius R can be computed as twice the integral

$$\int_M \sqrt{R^2-(x^2+y^2)}\ \mathrm{d}x\mathrm{d}y$$

where $M=\{(x,y)\in\mathbb{R}^2, x^2+y^2\le R^2\}$. We have

$$\int_M \sqrt{R^2-(x^2+y^2)}\ \mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_0^R \sqrt{R^2-r^2}r\ \mathrm{d}r\mathrm{d}\alpha = 2\pi\int_0^R \sqrt{R^2-r^2}r\ \mathrm{d}r$$

and we use a (one-dimensional) substitution $t=R^2-r^2$. In that case $\mathrm{d}t=-2r\ \mathrm{d}r$ and we have

$$2\pi\int_0^R \sqrt{R^2-r^2}r\ \mathrm{d}r = -\pi\int_{R^2}^0 \sqrt{t}\ \mathrm{d}t = \pi\int_0^{R^2}\sqrt{t}\ \mathrm{d}t = \pi\left[\frac{t^{3/2}}{\frac{3}{2}}\right]_0^{R^2} = \pi\frac{2}{3}R^3.$$

Note that this is just one half of the demanded volume. Therefore, we have just deduced the well known relation

$$V=\frac{4}{3}\pi R^3.$$

- Let compute an area of the set M which is given by the following conditions:

$$(x^2+y^2)^2\le 2xy,\ x\ge 0,\ y\ge 0.$$

We use the polar coordinates. The second and third condition yields $\alpha\in[0,\pi/2]$. Next, we plug the polar coordinates into the first condition to deduce

$$r^4\le 2r^2\cos\alpha\sin\alpha$$

and since $r>0$, we may divide the inequality by r to get

$$r^2\le 2\cos\alpha\sin\alpha$$

and thus (recall that $\sin(2\alpha)=2\sin\alpha\cos\alpha$)

$$r\le\sqrt{\sin(2\alpha)}.$$

We deduce that

$$\int_M 1\ \mathrm{d}x\mathrm{d}y = \int_0^{\pi/2}\int_0^{\sqrt{\sin(2\alpha)}} r\ \mathrm{d}r\mathrm{d}\alpha = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{\sin(2\alpha)}}\mathrm{d}\alpha = \frac{1}{2}\int_0^{\pi/2}\sin(2\alpha)\ \mathrm{d}\alpha$$
$$= -\frac{1}{4}\left[\cos(2\alpha)\right]_0^{\pi/2} = \frac{1}{2}$$

- Let compute an area of an ellipse which is given as

$$M=\left\{(x,y)\in\mathbb{R}^2,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$$

for some positive reals a and b.
We define

$$x = ar\cos\alpha =: \varphi(r,\alpha)$$
$$y = br\sin\alpha =: \psi(r,\alpha)$$

It holds that $\Phi^{-1}(M) = (0,1) \times (0,2\pi)$. Furthermore, we have

$$J\Phi(r,\alpha) = \begin{pmatrix} a\cos\alpha & -ar\sin\alpha \\ b\sin\alpha & br\cos\alpha \end{pmatrix}$$

and therefore $\det J\Phi(r,\alpha) = abr$. We get

$$\int_M 1 \ \mathrm{d}x\mathrm{d}y = \int_{(0,1)\times(0,2\pi)} abr \ \mathrm{d}r\mathrm{d}\alpha = \int_0^1 abr\mathrm{d}r \int_0^{2\pi} 1 \ \mathrm{d}\alpha = \pi ab.$$

- **The Laplace integral:** We use polar coordinates in order to deduce the value of integral

$$\int_0^\infty e^{-ax^2} \ \mathrm{d}x, \ \ a > 0.$$

Note that this integral cannot be evaluated by 'standard' one-dimensional methods. We denote $I := \int_0^\infty e^{-ax^2} \ \mathrm{d}x$ and $M = (0,\infty) \times (0,\infty)$. We have

$$\int_M e^{-ax^2-ay^2} \ \mathrm{d}x\mathrm{d}y = \int_0^\infty e^{-ax^2} \ \mathrm{d}x \int_0^\infty e^{-ay^2} \ \mathrm{d}y = I^2.$$

We use polar coordinates, i.e.

$$\begin{aligned} x &= r\cos\alpha \\ y &= r\sin\alpha \end{aligned}$$

Note that $\Phi((0,\infty) \times (0,\pi/2)) = M$. Therefore

$$\int_M e^{-a(x^2+y^2)} \ \mathrm{d}x\mathrm{d}y = \int_0^{\pi/2} \int_0^\infty e^{-ar^2} r \ \mathrm{d}r\mathrm{d}\alpha = \frac{\pi}{2} \int_0^\infty e^{-ar^2} r \ \mathrm{d}r$$

and we use a (one-dimensional) substitution $r^2 = t$ to get

$$\frac{\pi}{2} \int_0^\infty e^{-ar^2} r \ \mathrm{d}r = \frac{\pi}{4} \int_0^\infty e^{-at} \ \mathrm{d}t = -\frac{\pi}{4}\frac{1}{a} \left[e^{-at}\right]_0^\infty = \frac{\pi}{4a}.$$

We have just deduced that

$$I = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Index

bounded set, 6

cross-section, 23

global extreme, 3

Integration by parts, 13

Jacobian determinant, 27
Jacobian matrix, 27

Lagrange multipliers, 7
local extreme, 3

Method of substitution, 13

polar coordinates, 29
primitive function, 11

regular mapping, 27
Riemann integral, 18

stationary point, 3