Math, Functions

Václav Mácha

University of Chemistry and Technology

Basic notions

Definition of a function

Let $f \subset (X \times Y)$ be such that
$\forall x \in X,\ \forall y_1, y_2 \in Y,\ ((\langle x, y_1\rangle \in f)\&(\langle x, y_2\rangle \in f)) \Rightarrow (y_1 = y_2)$.

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A domain is a set of all $x \in X$ for which there exists y such that $f(x) = y$. The domain of f is denoted by $\mathrm{Dom}\, f$. The set of all $y \in Y$ for which there exists $x \in X$ such that $f(x) = y$ is called range and it is denoted by $\mathrm{Ran}\, f$.

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Examples: $\{\langle 1, 1\rangle, \langle 1, 2\rangle\}$ is not a function – the 'input' 1 has two possible outputs 1 and 2.
The set $f = \{\langle 1, 1\rangle, \langle 2, 0\rangle, \langle 3, 5\rangle\}$ is a function. It can be also written as $f(1) = 1$, $f(2) = 0$ and $f(3) = 5$. It holds that $\mathrm{Dom}\, f = \{1, 2, 3\}$ and $\mathrm{Ran}\, f = \{0, 1, 5\}$.

Let $A \subset \mathrm{Dom}\, f$. An image of A (denoted by $f(A)$) is a set defined as $f(A) = \{y \in \mathrm{Ran}\, f,\ \exists x \in A,\ y = f(x)\}$
Let $B \subset \mathrm{Ran}\, f$. A preimage of B (denoted by $f^{-1}(B)$) is a set defined as $f^{-1}(B) = \{x \in \mathrm{Dom}\, f,\ \exists y \in B,\ y = f(x)\}$
Mention, please, that f^{-1} is still undefined (it will be done in a few minutes). In particular, $f^{-1}(B)$ has different meaning than $f^{-1}(y)$, $y \in \mathrm{Ran}\, f$.

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Proof: It holds that

$$\begin{aligned}(y \in f(A \cup B)) &\Rightarrow (\exists x \in (A \cup B),\ y = f(x)) \\ &\Rightarrow ((\exists x \in A,\ y = f(x)) \vee (\exists x \in B,\ y = f(x))) \\ &\Rightarrow ((y \in f(A)) \vee (y \in f(B))) \Rightarrow (y \in f(A) \cup f(B))\end{aligned}$$

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On the other hand

$$\begin{aligned}(y \in f(A) \cup f(B)) &\Rightarrow ((y \in f(A)) \vee (y \in f(B)))\\ &\Rightarrow ((\exists x \in A,\ y = f(x)) \vee (\exists x \in B,\ y = f(x)))\\ &\Rightarrow (\exists x \in (A \cup B),\ y = f(x)) \Rightarrow (y \in f(A \cup B))\end{aligned}$$

which yields $(f(A) \cup f(B)) \subset f(A \cup B)$. This concludes the proof.

A function $f : X \to Y$ is said to be

- injective if $\forall x_1, x_2 \in \mathrm{Dom}\, f,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. (one-to-one)
- surjective, if $\mathrm{Ran}\, f = Y$, (onto)
- bijective, if it is surjective and injective.

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Example: a function $f : \{1, 2\} \to \{1\}$, $f(1) = 1$, $f(2) = 1$ is not injective (there are two arguments giving the same value), however, it is surjective. a function $f : \{1, 2\} \mapsto \{1, 2, 3\}$, $f(1) = 1$, $f(2) = 3$ is injective, but not surjective (there is no argument giving the number 2).

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Note that if $f : X \to Y$ is bijective, then $\mathrm{Dom}\, f$ has the same number of elements as Y (both sets have the same cardinality – for example, the sets of all natural numbers and of all even natural numbers have the same size).

Let $f : X \to Y$, $g : Y \to Z$ be such that $\mathrm{Ran}\, f \subset \mathrm{Dom}\, g$. Then we define a composition of f and g, $f \circ g : X \to Z$ as

$$g \circ f(x) = g(f(x)).$$

For example, take $f : \{1,2\} \to \{1,2,3\}$, $f(1) = 2$, $f(2) = 3$ and $g : \{1,2,3\} \to \{1,2\}$, $g(1) = 2$, $g(2) = 1$, $g(3) = 2$. Then

$$g \circ f : \{1,2\} \to \{1,2\}, g \circ f(1) = 1,\ g \circ f(2) = 2$$

and

$$f \circ g : \{1,2,3\} \to \{1,2,3\}, f \circ g(1) = 3,\ f \circ g(2) = 2,\ f \circ g(3) = 3.$$

A function $f : X \to X$, $f(x) = x$ is called identity.
Let $f : X \to Y$ be arbitrary. If there is $g : Y \to X$ such that $g \circ f(x) = x$ then g is called an inverse function to f and f is called an invertible function.

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Observation: Let $f : X \to Y$, $\mathrm{Dom}\, f = X$. Then f is invertible iff f is injective.

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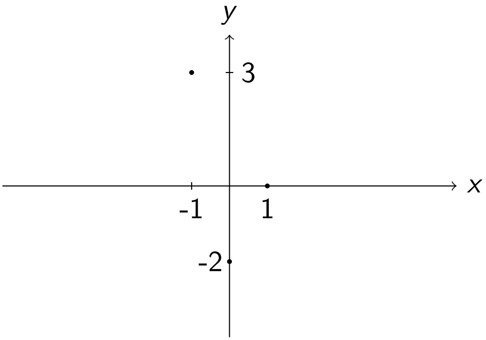
Observation: Let $f : X \to Y$, $\mathrm{Dom}\, f = X$. Then f is invertible iff f is injective.
Let f be injective. Then $\forall y \in \mathrm{Ran}\, f\ \exists x \in X$ such that $y = f(x)$. It suffices to define $f^{-1}(y) = x$.
Let f be not injective. There exists $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2) = y$. Let $f^{-1}(y) = x_1$ – this is necessary to have $f^{-1}(f(x_1)) = x_1$. But then $f^{-1}(f(x_2)) = f^{-1}(y) = x_1 \neq x_2$ and f^{-1} is not an inverse function.

Let $A \subset X$. A function $f : X \to \{0, 1\}$ is called an indicator function if $f(x) = 1$ if $x \in A$ and $f(x) = 0$ if $x \notin A$. Such function is denoted by χ_A.

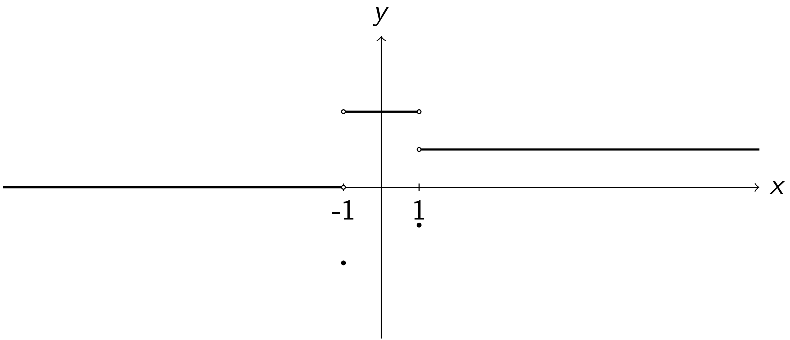
A function $f : X \mapsto \mathbb{R}$ is bounded from above if there is $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in \mathrm{Dom}\, f$. It is bounded from below if there is $m \in \mathbb{R}$ such that $f(X) \geq m$ for all $x \in \mathrm{Dom}\, f$. We say that f is bounded if it is bounded from below and from above.

Real functions

We turn our attention to real functions, i.e., functions $f : \mathbb{R} \to \mathbb{R}$.
A *graph* of such function is a subset of plane consisting of point $\langle x, f(x) \rangle$.
For example, the graph of a function $f = \{\langle 1, 0\rangle, \langle -1, 3\rangle, \langle 0, -2\rangle\}$ is the following



A graph of function $f = 2\chi_{(-1,1)} - 2\chi_{\{-1,1\}} + \chi_{[1,\infty)}$ is

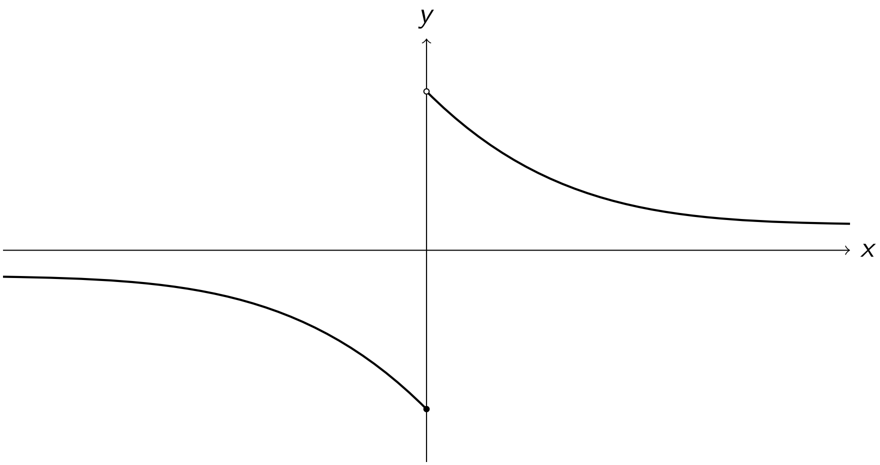


Properties
Monotonicity

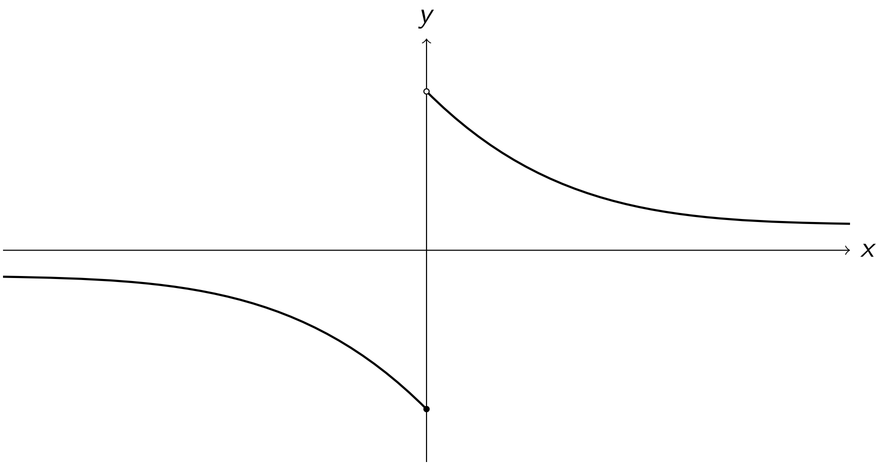
Let $f : \mathbb{R} \to \mathbb{R}$ and let $I \subset \mathrm{Dom}\, f$. We say that f is on I

- *increasing*, if $\forall x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$,
- *decreasing*, if $\forall x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$,
- *non-decreasing*, if $\forall x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$,
- *non-increasing*, if $\forall x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

If f posses one of these properties we say that f is *monotone*.



This function is decreasing on an interval $(-\infty, 0]$ and it is decrasing on $(0, \infty)$.



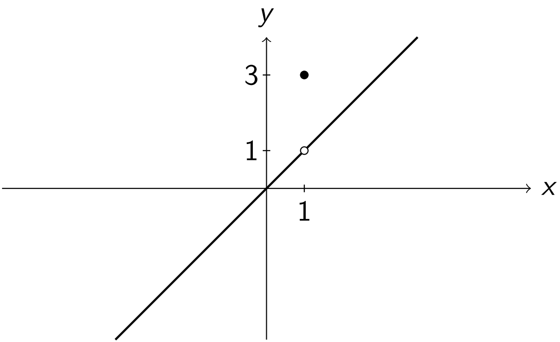
This function is decreasing on an interval $(-\infty, 0]$ and it is decrasing on $(0, \infty)$. However, it is not monotone on whole $\mathbb{R}$. Indeed, it is enough to take $x_1 = -1$ and $x_2 = 1$. Clearly $f(x_1) < f(x_2)$ and the function may not be decreasing (even non-increasing).

Continuity

A function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to be *continuous* at point $x_0 \in \mathrm{Dom}\, f$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (x_0 - \delta, x_0 + \delta) \cap \mathrm{Dom}\, f, \ |f(x) - f(x_0)| < \varepsilon.$$

Take function $f(x) = x\chi_{\mathbb{R}\setminus\{1\}} + 3\chi_{\{1\}}$. Its graph is



This function is certainly continuous for every $x \in (-\infty, 1) \cap (1, \infty)$. However it is discontinuous at $x = 1$.

A function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to be *left-continuous* (resp. *right continuous*) at a point $x_0 \in \mathrm{Dom}\, f$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (x_0 - \delta, x_0) \cap \mathrm{Dom}\, f,\ |f(x) - f(x_0)| < \varepsilon$$

$$(\text{resp. } \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (x_0, x_0 + \delta) \cap \mathrm{Dom}\, f,\ |f(x) - f(x_0)| < \varepsilon)$$

Further, we say that f is continuous on a set $S \subset \mathbb{R}$ if it is continuous at all of its points.

Observation: Let f and g be functions continuous at x_0. Then $f \pm g$ and $f \cdot g$ are also continuous at x_0. Moreover, if $g(x_0) \neq 0$ then also $\frac{f}{g}$ is continuous at x_0.

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Proof: We prove it for $f + g$ as $f - g$ can be done similarly. Due to continuity we have $\forall \varepsilon > 0\ \exists \delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ and $|g(x) - g(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta$. But this means that (due to the triangle inequality)

$$|f(x) + g(x) - (f(x_0) + g(x_0))| < |f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon.$$

Now we turn our attention to the product rule. First of all, since $f(x_0)$ is real and the function is continuous, there exists $\delta_1 > 0$ and $M_1 > 0$ such that $|f(x)| < M_1$ whenever $x \in (x_0 - \delta_1, x_0 + \delta_1) \cap \mathrm{Dom}\, f$ (see exercises at the end of this lecture). Similarly, there exists $\delta_2 > 0$ and $M_2 > 0$ such that $|g(x)| < M_2$ whenever $x \in (x_0 - \delta_2, x_0 + \delta_2) \cap \mathrm{Dom}\, f$. Due to continuity, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2M_2}$ and $|g(x) - g(x_0)| < \frac{\varepsilon}{2M_1}$ for all $x \in (x_0 - \delta, x_0 + \delta)$. We may moreover assume that $\delta < \min\{\delta_1, \delta_2\}$. Then we have

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)| < \varepsilon \end{aligned}$$

for all $x \in (x_0 - \delta, x_0 + \delta)$.

To prove the last claim it suffices to show that $\frac{1}{g}$ is continuous at x_0 and to use the just proven product rule. Without loss of generality, assume that $g(x_0) > 0$ and denote its value by $y_0 = g(x_0)$. Then, due to the continuity of g, there exists $\delta_1 > 0$ such that $g(x) > \frac{y_0}{2}$ for all $x \in (x_0 - \delta_1, x_0 + \delta_1) \cap \mathrm{Dom}\, f$. Further, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|g(x) - g(x_0)| < y_0^2 \frac{\varepsilon}{2}$ for each $x \in (x_0 - \delta, x_0 + \delta)$ and, moreover, we assume that $\delta < \delta_1$. Then we have

$$\left| \frac{1}{g(x)} - \frac{1}{g(x_0)} \right| = \left| \frac{g(x_0) - g(x)}{g(x)g(x_0)} \right| \leq \frac{|g(x_0) - g(x)|}{y_0 \frac{y_0}{2}} < \varepsilon$$

for each $x \in (x_0 - \delta, x_0 + \delta) \cap \mathrm{Dom}\, f$.
The proof is complete.

Parity and periodicity

Let $f : \mathbb{R} \to \mathbb{R}$ fulfill $\forall x \in \mathrm{Dom}\, f,\ -x \in \mathrm{Dom}\, f$. Then we say that

- f is *odd* if $f(-x) = -f(x)$,
- f is *even* if $f(-x) = f(x)$.

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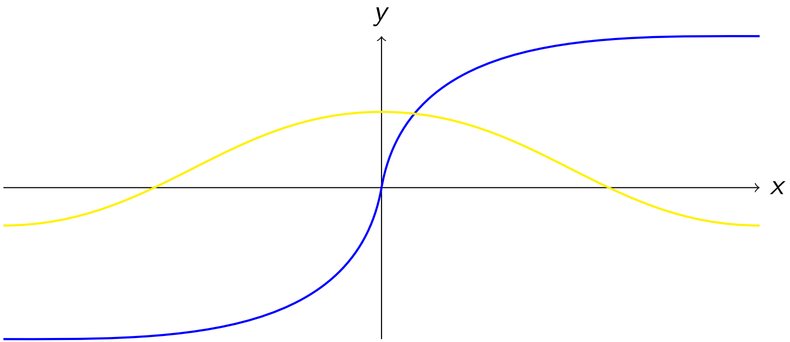

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{Dom}\, f = \mathbb{R}$ is called *periodic*, if there is a number $l > 0$ such that $f(x) = f(x + l)$ for all $x \in \mathbb{R}$. The least number l with that property is called a *period* of a function f and f is then $l-$periodic.

We introduce a notion of a maximum and minimum of set $A \subset \mathbb{R}$.

Definition

Let $\sup A$ be an element of $A \subset \mathbb{R}$. Then $\sup A$ is the highest number of A (or a maximum of A) and we write $\sup A = \max A$. Similarly, if $\inf A$ is an element of A, then $\inf A$ will be the lowest number of A (or a minimum of A) and we write $\inf A = \min A$.

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The minimum and maximum does not necessarily exists for a general set $A \subset \mathbb{R}$. For example, $A = \left\{\frac{1}{n},\ n \in \mathbb{N}\right\}$ has maximum 1, however, minimum does not exists. The infimum 0 is not contained in this set.

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Definition

Let f be continuous on an interval $I \subset \mathbb{R}$. Then we write $f \in \mathcal{C}(I)$.

Theorem (Weierstrass)

Let $f \in \mathcal{C}([a, b])$. Then f is bounded and there exists $t, u \in [a, b]$ such that $f(u) \leq f(x) \leq f(t)$ for all $x \in [a, b]$.

Actually, the previous theorem states that every function which is continuous on a closed interval attains its maximum and minimum value.

Theorem (Bolzano)

Let $f \in \mathcal{C}([a,b])$ *and* $f(a)f(b) < 0$. *Then there is* $\eta \in (a,b)$ *such that* $f(\eta) = 0$.

Theorem (Bolzano)

Let $f \in \mathcal{C}([a,b])$ and $f(a)f(b) < 0$. Then there is $\eta \in (a,b)$ such that $f(\eta) = 0$.

One can then deduce that every continuous function has the Darboux property (or intermediate value property):

Definition

A function $f : \mathbb{R} \to \mathbb{R}$, whose $\mathrm{Dom}\, f$ is an interval, is said to have the Darboux property if for every $x, y \in \mathrm{Dom}\, f$ and every $\tau \in (f(x), f(y))$ there exists $\varphi \in (x, y)$ such that $\tau = f(\varphi)$.

Lemma

Let f be an odd function and $(-a, a) \subset \operatorname{Dom} f$ for some $a > 0$. Then $f(0) = 0$.

Elementary functions

Polynomials are function which arises from a constant function $f \equiv c$, $c \in \mathbb{R}$ and an identity function $f(x) = x$ by finite number of multiplication and additions. In particular, every polynomial is of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0,$$

where $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$. The numbers $a_0, \ldots, a_n$ are called coefficients. The degree of $p(x)$ is n in a case $a_n \neq 0$ and we write $\operatorname{Deg} p = n$. The term $a_n x^n$ is called a *leading term*. Recall that $p(x) = x^n$ is odd function for odd n and it is an even function for n even. The maximal domain of $p(x)$ is always $\mathbb{R}$. All x such that $p(x) = 0$ are called *roots* of polynomial p. Let x_0 be a root of $p(x)$. Then $p(x) = (x - x_0)q(x)$ where $q(x)$ is a polynomial and it holds that $\operatorname{Deg} p(x) = \operatorname{Deg} q(x) + 1$.

A *rational function* is a fraction whose numerator and denominator are polynomials. I.e., a rational function f is of the form

$$f(x) = \frac{p(x)}{q(x)}.$$

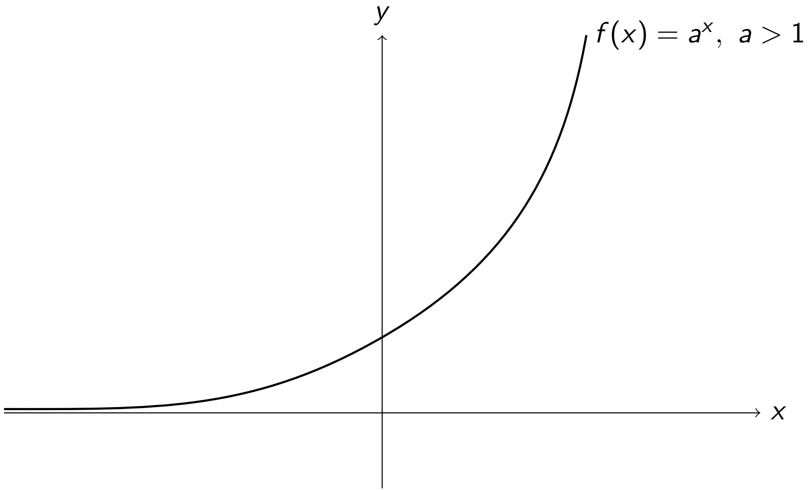
The domain of f is all real numbers except roots of $q(x)$.

Exponential function

Consider a number $a > 0$. Let $n \in \mathbb{N}$, we define $a^n = a \cdot a \cdot \ldots \cdot a$ where a appears n times on the right hand side. Further, we define $a^{\frac{1}{n}}$ as such number b that $b^n = 1$. This allows to define a^r for all rational numbers $r \in Q$ (do not forget $a^{-r} = \frac{1}{a^r}$). Namely, let $r > 0$, we define $a^r = a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}}$. For $r < 0$ we take $a^r = \frac{1}{a^{-r}}$. Finally, we are allowed to define uniquely a continuous function

$$f(x) = a^x \tag{1}$$

whose values are prescribed in the aforementioned way. Since the function is constant for $a \equiv 1$, we remove this value from our definition and we consider the relation (1) only for $a \in (0,1) \cup (1,\infty)$. It holds that $\mathrm{Dom}\, f = \mathbb{R}$ and $\mathrm{Ran}\, f = (0,\infty)$. Further, $f(0) = 1$ (roughly speaking, every number powered to 0 equals one). The function is strictly increasing for $a > 1$ and strictly decreasing for $a < 1$.


y
$f(x) = a^x, \ a > 1$
x

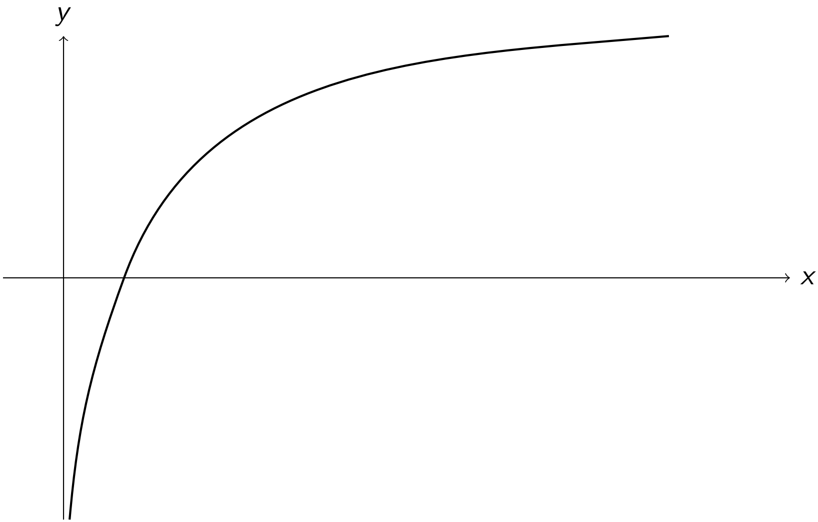
Logarithm

Since $x \mapsto a^x$ is injective there exists an inverse function. We will denote it by $\log_a$ and it is called logarithm to base a. In particular

$$\log_a y = x \quad \Leftrightarrow \quad a^x = y.$$

Recall that $a \in (0,1) \cup (1,\infty)$ and, due to the properties of the inverse functions, $\mathrm{Dom} \log_a = (0,\infty)$ and $\mathrm{Ran} \log_a = \mathbb{R}$. Recall also, that since $a^0 = 1$, we have $\log_a 1 = 0$ for every $a \in (0,1) \cup (1,\infty)$.

The graph of $f(x) = log_a(x)$, $a > 1$ is the following



Let e be Euler's number (this is an irrational number which will be defined later, its approximate value is 2.72). The logarithm to base e is called *natural logarithm* and, because of its importance, we omit the index e in its notation.

Next, we define *nth root* $f(x) = \sqrt[n]{x}$ as an inverse to $g(x) = x^n$. Recall that g is invertible for n odd and $\mathrm{Dom}\, g = \mathrm{Ran}\, g = \mathbb{R}$. Thus,
$\mathrm{Dom}\, \sqrt[n]{x} = \mathrm{Ran}\, \sqrt[n]{x} = \mathbb{R}$ for n odd.
However, g is not invertible for n even. In that case we have to restrict the domain of g to $[0, \infty)$ in order to have an injective function. The range of this restricted function is also $[0, \infty)$. As a consequence,
$\mathrm{Dom}\, \sqrt[n]{x} = \mathrm{Ran}\, \sqrt[n]{x} = [0, \infty)$ for n even.
The nth root is always an increasing function.

There is just one pair of continuous functions $s(x)$ and $c(x)$ with the following properties

- $s(x)^2 + c(x)^2 = 1$
- $s(x+y) = s(x)c(y) + c(x)s(y)$
- $c(x+y) = c(x)c(y) - s(x)s(y)$
- $0 < xc(x) < s(x) < x$ for all $x \in (0,1)$.

The function s is called sinus and the function c is called cosine. We also introduce notation $\sin x = s(x)$ and $\cos x = c(x)$. These functions have the following properties:

- $\mathrm{Dom}\sin x = \mathrm{Dom}\cos x = \mathbb{R}$, $\mathrm{Ran}\sin x = \mathrm{Ran}\cos x = [-1,1]$.
- $\sin x$ is an odd function, $\cos x$ is an even function.
- $\sin x$ and $\cos x$ are 2π periodic function.

There are several 'known' values of sin and cos:

$x =$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0

Besides, we define a function $\tan x = \frac{\sin x}{\cos x}$ (tangens) and a function $\cot x = \frac{\cos x}{\sin x}$ (cotangens). These functions are $\pi-$periodic, their range is $\mathbb{R}$ and

$$\mathrm{Dom}\ \tan x = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi,\ \ k \in \mathbb{Z} \right\},\ \mathrm{Dom}\ \cot x = \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}.$$

Roughly speaking, cyclometric functions are inverse functions to the aforementioned trigonometric functions. However, every trigonometric function is periodic and thus it is not one-to-one. To obtain the inverse function, we have to restrict the domain of every trigonometric function. In particular, we define functions $\sin_r$, $\cos_r$, $\tan_r$ and $\cot_r$ as follows

$$\sin_r x = \sin x,\ \mathrm{Dom}\ \sin_r = [-\pi_2, \pi_2]$$
$$\cos_r x = \cos x,\ \mathrm{Dom}\ \cos_r = [0, \pi]$$
$$\tan_r x = \tan x,\ \mathrm{Dom}\ \tan_r = [-\pi_2, \pi_2]$$
$$\cot_r x = \cot x,\ \mathrm{Dom}\ \cot_r = [0, \pi]$$

Now, since these functions are injective, we may define

$$\arcsin = \sin_r^{-1}$$
$$\arccos = \cos_r^{-1}$$
$$\arctan = \tan_r^{-1}$$
$$\mathrm{arccot} = \cot_r^{-1}$$

Let write down several properties of each function:

- $\mathrm{Dom}\ \mathrm{arcsin} = [-1,1]$, $\mathrm{Ran}\ \mathrm{arcsin} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, arcsin is an increasing function and $\mathrm{arcsin}(-1) = -\frac{\pi}{2}$, $\mathrm{arcsin}(0) = 0$ and $\mathrm{arcsin}(1) = \frac{\pi}{2}$
- $\mathrm{Dom}\ \mathrm{arccos} = [-1,1]$, $\mathrm{Ran}\ \mathrm{arccos} = [0,\pi]$, arccos is a decreasing function and $\mathrm{arcsin}(-1) = \pi$, $\mathrm{arcsin}(0) = \frac{\pi}{2}$ and $\mathrm{arcsin}(1) = 0$.
- $\mathrm{Dom}\ \mathrm{arctan} = \mathbb{R}$, $\mathrm{Ran}\ \mathrm{arctan} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, arctan is an increasing function and $\mathrm{arctan}(0) = 0$.
- $\mathrm{Dom}\,\mathrm{arccot} = \mathbb{R}$, $\mathrm{Ran}\,\mathrm{arccot} = (0,\pi)$, arccot is a decreasing function and $\mathrm{arccot}\,(0) = \frac{\pi}{2}$.

Limits

Definition

A *limit point* of a set $S \subset \mathbb{R}$ is every point $x_0 \in \mathbb{R}$ such that for every $\delta > 0$ it holds that $((x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \cap S \neq \emptyset$.

Consider, for example, $S = (0,1) \cup \{2\}$. The set of all its limit point is a closed interval $[0,1]$.

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Consider, for example, $S = (0, 1) \cup \{2\}$. The set of all its limit point is a closed interval $[0, 1]$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ and let x_0 be a limit point of $\mathrm{Dom}\, f$. We say, that $A \in \mathbb{R}$ is a *limit of f at* x_0 if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in ((x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \cap \mathrm{Dom}\, f,\ |f(x) - A| < \varepsilon.$$

We write

$$\lim_{x \to x_0} f(x) = A$$

$\mathbb{R}^*$ modification:

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ and let x_0 be a limit point of $\mathrm{Dom}\, f$. We say that $\lim_{x \to x_0} = \infty$ if

$$\forall M > 0,\ \exists \delta > 0,\ \forall x \in ((x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)) \cap \mathrm{Dom}\, f,\ \ f(x) > M.$$

Further, we say that $\lim_{x \to x_0} f(x) = -\infty$ if $\lim_{x \to x_0} -f(x) = \infty$.

Definition

Lef $f : \mathbb{R} \to \mathbb{R}$ be defined at least on (c, ∞) for some $c > 0$. We say that $\lim_{x \to \infty} = A \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists C > c,\ \forall x \in (C, \infty),\ |f(x) - A| < \varepsilon.$$

Further, we say that $\lim_{x \to \infty} = \infty$ if

$$\forall M > 0,\ \exists C > c,\ \forall x \in (C, \infty),\ f(x) > M.$$

Observation

Once the limit exists, it is determined uniquely.

Proof.

Let $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} f(x) = B$ for some different $A, B \in \mathbb{R}$. Take $\varepsilon = \frac{1}{3}|B - A|$. According to the definition of a limit, there exists $\delta > 0$ such that $|f(x) - A| < \varepsilon$ and, simultaneously, $|f(x) - B| < \varepsilon$ for some $x \in (x_0 - \delta, x_0 + \delta)$. We use the triangle inequality to deduce

$$|A - B| = |A - f(x) + f(x) - B| \le |A - f(x)| + |f(x) - B| \le \frac{2}{3}|A - B|.$$

The case of infinite limits is done by an obvious modification. □

Observation

Let f be a function continuous in a limit point $x_0 \in \mathrm{Dom}\, f$. Then

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Proof.

Let $\varepsilon > 0$ be arbitrary. As f is continuous, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \varepsilon,\ x \in \mathrm{Dom}\, f$. But that is exactly that δ which suits the definition of a limit. □

Here we would like to emphasize that every elementary function from the previous chapter is continuous on its domain.

This is the first tool which allows a computation. For example

$$\lim_{x \to 3} x - 5 = -2.$$

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Consider for example a function $f(x) = \frac{x^2+4x+3}{x^2-1}$. This function is clearly not defined at points -1 and 1 and is continuous everywhere else. Anyway, we may compute

$$\lim_{x \to -1} \frac{x^2 + 4x + 3}{x^2 - 1} = \lim_{x \to -1} \frac{(x+1)(x+3)}{(x-1)(x+1)} = \lim_{x \to -1} \frac{x+3}{x-1} = -1$$

Definition

Let x_0 be a limit point of $\mathrm{Dom}\, f$. We say that $A \in \mathbb{R}$ is a *left-sided limit* of f at x_0 (resp. *right-sided limit* of f in x_0) if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (x_0 - \delta, x_0) \cap \mathrm{Dom}\, f,\ |f(x) - A| < \varepsilon.$$

(resp.

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (x_0, x_0 + \delta) \cap \mathrm{Dom}\, f,\ |f(x) - A| < \varepsilon.)$$

We write

$$\lim_{x \to x_0 -} f(x) = A \quad (\text{resp.} \quad \lim_{x \to x_0 +} f(x) = A).$$

The infinite limits are defined similarly.

Lemma (Arithmetic of limits)

Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ *and let* x_0 *be a limit point of* $\mathrm{Dom}\, f$ *and* $\mathrm{Dom}\, g$. *Let, moreover,* $c \in \mathbb{R}$. *Then*

$$\begin{aligned}
\lim_{x \to x_0} (f(x) \pm g(x)) &= \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x) \\
\lim_{x \to x_0} cf(x) &= c \lim_{x \to x_0} f(x) \\
\lim_{x \to x_0} (f(x)g(x)) &= \lim_{x \to x_0} f(x) \lim_{x \to x_0} g(x) \\
\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}
\end{aligned} \tag{2}$$

assuming the right hand side has meaning.

Indefinite values:

$$0 \cdot \infty,\ \frac{0}{0},\ \frac{\infty}{\infty},\ \infty - \infty,\ 1^{\infty},\ 0^{\infty}$$

Note that the arithmetic of limits holds also for the one-sided limits.

Let compute a limit $\lim_{x\to\infty} \frac{x-1}{x-2}$. According to arithmetic of limits $\lim_{x\to\infty} x-1=\infty$ and $\lim_{x\to\infty} x-2=\infty$. However, we cannot write that

$$\lim_{x\to\infty} \frac{x-1}{x-2} = \frac{\infty}{\infty}$$

as we get an indefinite term. The trick here is to simplify by the most rapidly growing summand in the denominator:

$$\lim_{x\to\infty} \frac{x-1}{x-2} = \lim_{x\to\infty} \frac{1-\frac{1}{x}}{1-\frac{2}{x}} = \frac{1-0}{1-2\cdot 0} = 1.$$

Observation

Let $\lim_{x\to x_0} f(x) = A$ *for some* $x_0 \in \mathbb{R}$ *and* $A \in \mathbb{R}^*$*. Then also* $\lim_{x\to x_0-} f(x) = A$ *and* $\lim_{x\to x_0+} f(x) = A$*.*

Let consider $\lim_{x\to 0} \frac{1}{x}$. We are going to show that $\lim_{x\to 0-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0+} \frac{1}{x} = +\infty$. In such case, $\lim_{x\to 0} \frac{1}{x}$ does not exist according to the just mentioned observation.
Let $K > 0$. We take $\delta = \frac{1}{K}$ and, consequently, for all $x \in (0, \delta)$ it holds that $f(x) = \frac{1}{x} > \frac{1}{\delta} = K$ and $\lim_{x\to 0+} \frac{1}{x} = \infty$.
Similarly, for all $x \in (-\delta, 0)$ it holds that $f(x) = \frac{1}{x} < \frac{1}{\delta} = -K$ and thus $\lim_{x\to 0-} \frac{1}{x} = -\infty$.

Few exercises

$$\lim_{x\to 2} \frac{x^3 + x - 2}{x^2 + 1}$$

$$\lim_{x\to 2} \frac{x^3 + 3x - 14}{x^2 - 4x + 4}$$

$$\lim_{x\to -2} \frac{x^3 + 4x^2 - 8}{x^2 + 5x + 6}$$

$$\lim_{x\to \infty} \frac{x^4 - 5x}{x^2}$$

$$\lim_{x\to 1} \frac{x + 3}{x^2 - 2x + 1}$$

- $\lim_{x\to\infty} a^x = \infty$ for $a > 1$,
- $\lim_{x\to\infty} \log_a x = \infty$ for $a > 1$,
- $\lim_{x\to 0+} \log_a x = -\infty$ for $a > 1$,
- $\lim_{x\to\frac{\pi}{2}-} \tan x = \infty$,
- $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$,
- $\lim_{x\to\infty} \operatorname{arccot} x = 0$,
- $\lim_{x\to-\infty} \operatorname{arccot} x = \pi$.

The following limits are used without any further proofs:

- There is a number e such that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

  We recall that e is the Euler number (the base of natural logarithm) whose value is approx. 2.72.

The following limits are used without any further proofs:

- There is a number e such that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

  We recall that e is the Euler number (the base of natural logarithm) whose value is approx. 2.72.

- Further,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

- Finally,

$$\lim_{x\to 0} \frac{\log(x+1)}{x} = 1.$$

Lemma (Limit of composed function)

Let $\lim_{x\to x_0} g(x) = A$ *and* $\lim_{y\to A} f(y) = B$. *Then*

$$\lim_{x\to x_0} f(g(x)) = B,$$

if at least one of the following is true:

1. f *is continuous at the point* A *or*
2. *there is* δ *such that for all* $x \in (x - \delta, x_0) \cap (x_0, x + \delta)$ *it holds that* $g(x) \neq A$.

Let compute

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{4\left(\frac{x}{2}\right)^2}$$

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$$= \lim_{x\to 0} \frac{2\sin^2\left(\frac{x}{2}\right)}{4\left(\frac{x}{2}\right)^2}$$

Now we are allowed to use the Lemma LOCF, note that $g(x) = \frac{x}{2}$ is injective and thus the assumptions of LOCF are fulfilled.

Let compute

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{4\left(\frac{x}{2}\right)^2}$$

Now we are allowed to use the Lemma LOCF, note that $g(x)=\frac{x}{2}$ is injective and thus the assumptions of LOCF are fulfilled. Thus

$$\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{4\left(\frac{x}{2}\right)^2}=\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}$$

$$\overset{AL}{=}\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\overset{LOCF}{=}\frac{1}{2}.$$

Lemma (Sandwich Lemma)

Let $x_0 \in \mathbb{R}$ *and let there is* $\delta > 0$ *such that*

$$f(x) \leq g(x) \leq h(x), \ \forall x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta).$$

Then $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = A$ *implies* $\lim_{x \to x_0} g(x) = A$.

Let compute $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$. It holds that

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$$

for all x in, say, $(-1, 0) \cup (0, 1)$. Further,

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0.$$

Exercise:

- $\lim_{x\to\infty} \sin x$
- $\lim_{x\to\infty} \frac{\sin x}{x}$
- $\lim_{x\to 0} \frac{\sin(2x)}{e^x-1}$
- $\lim_{x\to\infty} \frac{(x+1)^4}{(x+\sqrt{x})^3}$
- $\lim_{x\to\infty} \sqrt{x}\left(\sqrt{2x}-\sqrt{2x-1}\right)$
- $\lim_{x\to 1} \left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$
- $\lim_{x\to\infty} \frac{(\sqrt{x^2+1}+x)^2}{\sqrt[3]{x^6+1}}$
- $\lim_{x\to 0} \frac{\tan x-\sin x}{x^3}$
- $\lim_{x\to 0} \frac{\sin(2x)}{\sqrt{x+3}-\sqrt{3}}$

Relation between a limit and continuity

Recall:

Lemma

A function $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x_0 \in Dom f$ if and only if $\lim_{x\to x_0} f(x) = f(x_0)$.

Exercises

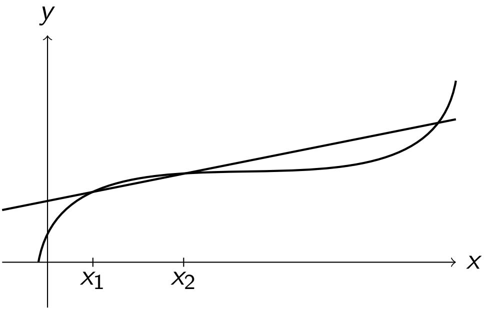
- We saw that $f(x) = x\chi_{\mathbb{R}\setminus\{1\}} + 3\chi_{\{1\}}$ is not continuous.
- Decide about the continuity of $f(x) = \left(\frac{1}{x}\right)\chi_{[1,\infty)} + \left(\frac{(2x+2)(x-1)}{(x+2)(x-1)}\right)\chi_{(-\infty,1)}$.
- How about the continuity of $f(x) = e^x\chi_{(-\infty,0]} + \left(\frac{\sin(4x)-\sin(3x)}{4x-3x}\right)\chi_{(0,\infty)}$

Let recall few facts of lines. Let have a line passing through two points $A = \langle a_1, a_2 \rangle$ and $B = \langle b_1, b_2 \rangle$ with $a_1 \neq b_1$. Then the slope of the line is a number $k = \frac{a_2 - b_2}{a_1 - b_1}$. The equation of the line has form

$$y = kx + q$$

where $q \in \mathbb{R}$ is determined such that the equation holds true for $y = a_2$ and $x = a_1$ (resp. $y = b_2$ and $y = b_1$).

Consider a graph of a function $f(x)$, for example, of the following form



The equation of the line passing through point $\langle x_1, f(x_1)\rangle$ and $\langle x_2, f(x_2)\rangle$ is

$$y = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1).$$

How to make a tangent line? Just simply tend with x_2 to x_1. So the tangent line has equation

$$y = k(x - x_1) + f(x_1)$$

where

$$k = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

assuming the limit exists.

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where

$$k = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

assuming the limit exists. We denote $h := x_2 - x_1$ and then we may write

$$k = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

Observation

Let $f'(x_0)$ is real. Then f is continuous at x_0.

Proof.

Indeed, it is enough to compute

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = f'(x_0) \cdot 0 = 0.$$

Consequently, $\lim_{x \to x_0} f(x) = f(x_0)$ and the function is continuous at x_0. □

Let $f : \mathbb{R} \to \mathbb{R}$. We define

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We say that f' is derivative of f.

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In particular, a derivative of f in a point x is a slope of the tangent line passing through $\langle x, f(x) \rangle$.
Let emphasize that f' does not exist for every function.

Exercise: Compute derivatives for:

- $f(x) = x^n,\ n \in \mathbb{N}$
- $f(x) = e^x$
- $f(x) = \sin x$
- $f(x) = \cos x$
- $f(x) = \log x$

To sum up:

$f(x)$	$f'(x)$	conditions
x^n	nx^{n-1}	$n \in \mathbb{N}$, $x \in \mathbb{R}$
e^x	e^x	$x \in \mathbb{R}$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\log x$	$\frac{1}{x}$	$x \in (0, \infty)$

Lemma

Let f and g be differentiable functions. Then

$$
\begin{aligned}
(f(x) \pm g(x))' &= f'(x) \pm g'(x) \\
(f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\
\left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}
$$

if both sides have sense.

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$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

if both sides have sense.

Exercise

- Compute $\left(x^5 - 4x^3 + \log x\right)'$.
- Compute $\left(x^3 \sin x\right)'$.
- Compute $\left(\frac{xe^x}{\cos x}\right)'$.

Exercise

- Compute $(\tan x)'$.

Lemma

Let f and g be differentiable functions and let $b = f(a)$. Then

$$(g \circ f)'(a) = g'(b)f'(a) = g'(f(a))f'(a).$$

Lemma

Let f and g be differentiable functions and let $b = f(a)$. Then

$$(g \circ f)'(a) = g'(b)f'(a) = g'(f(a))f'(a).$$

Exercise: Compute

- $\left(e^{2x}\right)'$
- $(5^x)'$ (and generally $(a^x)'$)
- $\left(\cos(x^2)\right)'$
- $\left(x^2\sqrt{x+1}\right)'$
- $(\arctan x)'$ (hint: use the fact that $x = \arctan \circ \tan x$)

To sum up, we present the following table:

$f(x)$	$f'(x)$	conditions
x^n	nx^{n-1}	$n \in \mathbb{R}$, x as usual
e^x	e^x	$x \in \mathbb{R}$
a^x	$\log a\ a^x$	$a \in (0,1) \cup (1,\infty)$, $x \in \mathbb{R}$
$\log x$	$\frac{1}{x}$	$x \in (0,\infty)$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$
$\cot x$	$-\frac{1}{\sin^2 x}$	$x \in \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$
$\mathrm{arccot}\, x$	$-\frac{1}{1+x^2}$	$x \in \mathbb{R}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$x \in (-1,1)$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$x \in (-1,1)$

Exercises

- Write the equation of the tangent line to the graph of $f(x) = x^2 + 5x + 8$ at a point $x_0 = -2$, $y_0 = ?$.
- Find all tangent lines to the graph of $f(x) = x + \frac{1}{x^2}$ which are parallel to the line $y = -2x$.

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As a matter of fact, the formula for the tangent line is

$$y = f'(x_0)(x - x_0) + y_0$$

where x_0 and y_0 is the point of tangency.

Definition

We say that $f : \mathbb{R} \to \mathbb{R}$ attains its local maximum at a point $x_0 \in \mathrm{Dom}\, f$ if

$$\exists \delta > 0,\ \forall x \in (x_0 - \delta, x_0 + \delta) \cap \mathrm{Dom}\, f,\ \ f(x) \leq f(x_0).$$

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Let f be defined on an interval (a, b) let it attains its local maximum (resp. minimum) in a point $x_0 \in (a, b)$, and let $f'(x_0)$ exist. Then $f'(x_0) = 0$.

Example:

- Find all points where the function

$$f(x) = x^2 e^x$$

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Definition

The point x_0 for which $f'(x_0) = 0$ is called a *stationary point*.

Lemma

Let x_0 be a stationary point and let $f \in C^2$ (meaning: f has continuous second derivatives). Then

1. *if $f''(x_0) > 0$, the function has a local minimum at x_0,*
2. *if $f''(x_0) < 0$, the function has a local maximum at x_0,*
3. *if $f''(x_0) = 0$, we do not know anything.*

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Example:

- Finish the previous example, i.e., clasiffy the extremes of $f(x) = x^2 e^x$.

Definition

Maximum of $f : \mathbb{R} \to \mathbb{R}$ on $[a, b] \subset \mathbb{R}$ is attained in $x_0 \in [a, b]$ if $f(x_0) \geq f(x)$ for every $x \in [a, b]$. Similarly, minimum of f is attained in $x_1 \in [a, b]$ if $f(x_1) \leq f(x)$ for every $x \in [a, b]$.

Example:

- Find the maximum and minimum of

$$f(x) = 2x^3 - 3x^2 - 12x + 8 \quad \text{on } [-3, 3].$$

Lemma

Let $f \in C^1$ *and let* $[a, b] \subset \mathrm{Dom}\, f$.

1. *If* $f'(x) > 0$ *for every* $x \in (a, b)$, *then* f *is increasing on* $[a, b]$.
2. *If* $f'(x) < 0$ *for every* $x \in (a, b)$, *then* f *is decreasing on* $[a, b]$.
3. *If* $f'(x) \geq 0$ *for every* $x \in (a, b)$, *then* f *is non-decreasing on* $[a, b]$.
4. *If* $f'(x) \leq 0$ *for every* $x \in (a, b)$, *then* f *is non-increasing on* $[a, b]$.

Exercise

- Find local extremes of $f(x) = 12x^5 - 15x^4 - 40x^3 + 60$. Determine the maximal intervals of monotonicity.

Definition

We say that $f : \mathbb{R} \to \mathbb{R}$ is convex on a set $I \subset \operatorname{Dom} f$ if for all $x, y, z \in I$, $x < y < z$ it holds that

$$\frac{f(y) - f(x)}{y - x} < \frac{f(z) - f(y)}{z - y}.$$

We say that f is concave on I if $-f$ is convex on I.

Definition

We say that $x \in \mathbb{R}$ is a point of inflection of $f : \mathbb{R} \to \mathbb{R}$ if f is continuous at x and there is $\delta > 0$ such that one of the following appears

1. f is concave on $(x - \delta, x)$ and convex on $(x, x + \delta)$ or
2. f is convex on $(x - \delta, x)$ and concave on $(x, x + \delta)$.

Observation

Let $f \in \mathcal{C}(I)$ for some interval $I \subset \mathbb{R}$. Assume that $f''(x)$ exists for all $x \in I$.

1. *If $f''(x) > 0$ for all $x \in I$ then f is convex on I.*
2. *If $f''(x) < 0$ for all $x \in I$ then f is concave on I.*

Example

- Find the interval of convexity and concavity of $f(x) = \frac{1}{x^3} + \frac{1}{x^2}$, find its points of inflection.

Asymptotes:

Definition

Let $\lim_{x\to\infty} \frac{f(x)}{x} = k_+ \in \mathbb{R}$ and let $\lim_{x\to\infty} f(x) - k_+x = q_+$. Then an *asymptote at* ∞ is a line with equation $y = k_+x + q_+$.
Let $\lim_{x\to-\infty} \frac{f(x)}{x} = k_- \in \mathbb{R}$ and let $\lim_{x\to-\infty} f(x) - k_-x = q_-$. Then an *asymptote at* $-\infty$ is a line with equation $y = k_-x + q_-$.

Exercises:

- Find the asymptotes of $f(x) = e^x + x + 1$.
- Find the asymptotes of $f(x) = \frac{x^3-x^2}{x^2+1}$.

The course of a function Now we are ready to describe the problem of the course of function. The task 'examine the course of the following function' consists of the following sub-tasks:

1. To find out the domain, to determine whether the function is even, odd or periodic.
2. To find intersections with axes.
3. To examine the behavior of the function at the edges of the domain.
4. To derive function, to determine sets where the function is increasing and decreasing, to determine extremes.
5. To differentiate the function for the second time, to determine sets where the function is concave, convex, to determine points of inflection.
6. To sketch a graph of the function.

Exercise:

- Examine the course of $f(x) = \frac{x^2+3}{x-1}$.

Further exercises

- Examine the course of $f(x) = 3x^5 - 5x^3$.
- Examine the course of $f(x) = x^2 + \frac{1}{x^2}$.
- Examine the course of $f(x) = \frac{|x-1|}{x+2}$.
- Examine the course of $f(x) = (x-4)\sqrt[3]{x}$.
- Examine the course of $f(x) = 3 + \sin x \cos x$.

Lemma (l'Hospital)

Let f and g have finite derivatives for all $x \in (a, b) \subset \mathbb{R}$. Assume $g'(x) \neq 0$ and

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = A \in \mathbb{R}^*.$$

Let moreover one of the following is true:

1. $\lim_{x \to a+} f(x) = 0$ *and* $\lim_{x \to a+} g(x) = 0$ *or*
2. $\lim_{x \to a+} |g(x)| = \infty$.

Then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = A.$$

Obviously, the same true is also for $x \to b-$.

Compute:

1. $\lim_{x\to 2} \frac{x^2-4}{x^2-x-2}$
2. $\lim_{x\to 0} \frac{x \sin x}{1-\cos x^2}$
3. $\lim_{x\to \infty} \frac{x}{\sqrt{x^2+1}}$
4. $\lim_{x\to \frac{\pi}{4}-} \tan(2x) \log(\tan x)$
5. $\lim_{x\to 0} \left(\frac{x-1}{2x^2} - \frac{1}{x(e^{2x}-1)}\right)$
6. $\lim_{x\to 0} (\cos(3x))^{\frac{1}{x^2}}$

Exercise

- Examine the course of $f(x) = \frac{\log x}{x} + 1$.
- Examine the course of $f(x) = (x+2)e^{\frac{1}{x}}$.
- Examine the course of $f(x) = (x+3)e^{x-2}$.
- Examine the course of $f(x) = x\sqrt{1-x^2}$.

Definition (The Taylor polynomial)

Let f be $n-$times differentiable at point x_0. Then the polynomial of the form

$$T_{f,x_0,n}(x)$$

$$:= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}}{n!}(x - x_0)^n$$

$$= \sum_{j=0}^{n} \frac{f^{(i)}}{i!}(x - x_0)^i$$

is called the Taylor polynomial for f at point x_0 of degree n.

Example

- Write the fourth-degree Taylor polynomial for $f(x) = x \log x$ at point $x_0 = 1$.

Lemma

Assume that f is $(n+1)-$times differentiable at x_0. Let $x \in \mathbb{R}$ be arbitrary and let f is $(n+1)-$times differentiable on a closed interval I with edges at x_0 and x. Then there is ζ in between of x and x_0 such that

$$f(x) - T_{f,x_0,n}(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x - x_0)^{n+1}.$$

Example

- Approximate the value of $\arctan 0,8$ by the Taylor polynomial of degree 3.
- What is the biggest possible mistake we made in the approximation of $\arctan 0,8$?

Some further exercies

- How long does it take to double your investment if the interest is x percent? The rule of 70 (or 69, 68 or whatever).
- Use the third-degree Taylor polynomial in order to deduce the approximate value of $\sqrt[3]{30}$.
- Use the Taylor polynomial at $x_0 = 0$ to deduce the approximate value of e with an error not higher than 0.001.