Functions of multiple variable

Václav Mácha

University of Chemistry and Technology

Definition

Let $M \subset \mathbb{R}^n$ be a nonempty set. A *real function of n variables* defined on a set M is a mapping f which uniquely assigns a real number y to every pair $(x_1, \dots, x_n) \in M$. We use the notation

$$y = f(x_1, \dots, x_n).$$

To denote the function itself we use a notation $f : M \mapsto \mathbb{R}$. The set M is called the *domain* of f and we write $M = \mathrm{Dom}\ f$.

Usually, the function will be given only by its formula without any specific domain. In that case, we assume that the domain is a maximal set for which has the formula sense. For example, a function

$$f(x_1, x_2) = \log(x_1 + x_2)$$

is defined on a set

$$\mathrm{Dom}\ f = \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 + x_2 > 0\}.$$

Example

- Find (and sketch) a maximal set $M \subset \mathbb{R}^2$ of such pairs (x_1, x_2) for which the function

$$f(x_1, x_2) = \frac{1}{\sqrt{x_1^2 + x_2 - 1}}.$$

- Find (and sketch) the maximal domain of a function

$$f(x, y) = \sqrt{1 - \log(y - x^2)}.$$

Definition

Let $z = f(x, y)$ be a function of two variables. The *graph* of f is a set

$$\text{graph} f = \{(x, y, f(x, y) \in \mathbb{R}^3,\ (x, y) \in \mathrm{Dom}\ f\}.$$

Definition

Let $z = f(x, y)$ be a function of two variables. The *graph* of f is a set

$$\text{graph} f = \{(x, y, f(x, y) \in \mathbb{R}^3,\ (x, y) \in \mathrm{Dom}\ f\}.$$

Example

- Sketch a graph of $f(x, y) = -x - 2y + 3$.
- Sketch a graph of $f(x, y) = x^2 + y^2$.

Definition

A *contour line* C at height $z_0 \in \mathbb{R}$ is a set

$$\{(x, y) \in \mathbb{R}^2,\ f(x, y) = z_0\}.$$

Example

- Find contour lines at heights $z_0 = -2, -1, 0, 1, 2$ for a function

$$f(x, y) = \frac{x^2 + y^2}{2x}.$$

- Find contour lines at heights $z_0 = -2, -1, 0, 1, 2$ for a function

$$f(x, y) = (x + y) + |x + y|$$

Few words about algebra of function:

Sum, product and division is defined 'pointwisely'. Consider, for example, functions $f(x,y) = e^{xy}$ and $g(x,y) = \sqrt{1-x^2-y^2}$. Then

- $(f+g)(x,y) = e^{xy} + \sqrt{1-x^2-y^2}$,
- $(fg)(x,y) = e^{xy}\sqrt{1-x^2-y^2}$,
- $\frac{f}{g}(x,y) = \frac{e^{xy}}{\sqrt{1-x^2-y^2}}$. Beware, here we have to exclude from the domain all points where g equals zero.

Few words about algebra of function:

Sum, product and division is defined 'pointwisely'. Consider, for example, functions $f(x,y) = e^{xy}$ and $g(x,y) = \sqrt{1-x^2-y^2}$. Then

- $(f+g)(x,y) = e^{xy} + \sqrt{1-x^2-y^2}$,
- $(fg)(x,y) = e^{xy}\sqrt{1-x^2-y^2}$,
- $\frac{f}{g}(x,y) = \frac{e^{xy}}{\sqrt{1-x^2-y^2}}$. Beware, here we have to exclude from the domain all points where g equals zero.

Composition of functions: Let $f : M \mapsto \mathbb{R}^n$ (this means that there are n functions $f_i : M \mapsto \mathbb{R},\ i \in \{1,\dots,n\}$) and $g : \mathbb{R}^n \mapsto \mathbb{R}$. Then a *composition* is a function $h = g \circ f$ defined as

$$h(x,y) = g(f_1(x,y), f_2(x,y)).$$

Similarly, if $f : M \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ then $h = g \circ f$ is defined as $h(x,y) = g(f(x,y))$

Definition

Let $M \subset \mathbb{R}^n$ and $f : M \to \mathbb{R}$. Next, let $\varphi : I \to M$ is a curve ($I \subset \mathbb{R}$ is an interval). Then $f \circ \varphi$ is a *cross-section* of f.

Definition

Let $M \subset \mathbb{R}^n$ and $f : M \to \mathbb{R}$. Next, let $\varphi : I \to M$ is a curve ($I \subset \mathbb{R}$ is an interval). Then $f \circ \varphi$ is a *cross-section* of f.

Example

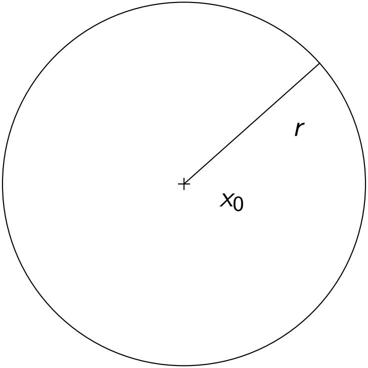
- What is the graph of a function
$$f(x, y) = (x + y)^2$$
on a line $p_a : (x, y) = (a, 0) + t(1, 1)$, $t \in \mathbb{R}$ for some $a \in \mathbb{R}$? And how about lines $q_b : (x, y) = (a, 0) + t(1, -1)$, $t \in \mathbb{R}$ for some $b \in \mathbb{R}$?.
- Draw a graph of a cross-section
$$f(x, y) = \frac{1}{x^2 + y^2}$$
along lines
$$(x, y) = t(\cos\alpha, \sin\alpha), t \in (0, \infty)$$
where $\alpha \in [0, 2\pi)$ is a parameter.

Topology

Definition

An open ball centered at $(x_0) \in \mathbb{R}^n$ with radius $r \in (0, \infty)$ is a set

$$B_r(x_0) = \{x \in \mathbb{R}^2, \|x - x_0\| < r\}.$$



Definition

A set $M \subset \mathbb{R}^2$ is *open* if for every $(x_0, y_0) \in \mathbb{M}$ there is $r > 0$ such that $B_r(x_0, y_0) \subset M$.
A set M is called *closed* if $\mathbb{R}^2 \setminus M$ is open.

Definition

A set $M \subset \mathbb{R}^2$ is *open* if for every $(x_0, y_0) \in \mathbb{M}$ there is $r > 0$ such that $B_r(x_0, y_0) \subset M$.
A set M is called *closed* if $\mathbb{R}^2 \setminus M$ is open.

Example

- A set $M := (0,1) \times (0,1)$ is open. Indeed, let $(a,b) \in M$. Define $\delta = \min\{a, b, 1-a, 1-b\}$. Since $a \in (0,1)$ and $b \in (0,1)$ we have $\delta > 0$. Necessarily, $B_{\delta/2}(a,b) \subset M$.
- On the other hand, a set $M := [0,1] \times (0,1)$ is not open. Consider for example a point $(1, 1/2) \in M$. Then every ball $B_r(1, 1/2)$ contains a point $(1 + r/2, 1/2)$ which is outside of M. Note that M is not closed. Why?

Few notes about open sets

- $\emptyset$ and $\mathbb{R}^n$ are open sets (and closed sets as well),
- a union of open sets is an open set,
- an intersection of two open sets is an open set.

Few notes about open sets

- $\emptyset$ and $\mathbb{R}^n$ are open sets (and closed sets as well),
- a union of open sets is an open set,
- an intersection of two open sets is an open set.

Let consider a set

$$M := \{(x, y), x \in (-1, 1),\ y < x^2\}.$$

Is this set open?

Few notes about open sets

- $\emptyset$ and $\mathbb{R}^n$ are open sets (and closed sets as well),
- a union of open sets is an open set,
- an intersection of two open sets is an open set.

Let consider a set

$$M := \{(x, y), x \in (-1, 1),\ y < x^2\}.$$

Is this set open?

Observation

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuous function. Then $f^{-1}(A)$ is an open set for every $A \subset \mathbb{R}$ open.

Few notes about open sets

- $\emptyset$ and $\mathbb{R}^n$ are open sets (and closed sets as well),
- a union of open sets is an open set,
- an intersection of two open sets is an open set.

Let consider a set

$$M := \{(x,y), x \in (-1,1),\ y < x^2\}.$$

Is this set open?

Observation

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuous function. Then $f^{-1}(A)$ is an open set for every $A \subset \mathbb{R}$ open.

Question

Few notes about open sets

- $\emptyset$ and $\mathbb{R}^n$ are open sets (and closed sets as well),
- a union of open sets is an open set,
- an intersection of two open sets is an open set.

Let consider a set

$$M := \{(x, y), x \in (-1, 1),\ y < x^2\}.$$

Is this set open?

Observation

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuous function. Then $f^{-1}(A)$ is an open set for every $A \subset \mathbb{R}$ open.

Question What is a *continuous function*? We will see later.

For now: A projection $p : \mathbb{R}^2 \mapsto \mathbb{R}$, $p(x, y) = x$ is a continuous function (as well as projection $q(x, y) = y$). A sum, difference and product of two continuous functions are continuous functions. A quotient of two continuous function is again a continuous function. A composition of two continuous function is a continuous function.

For now: A projection $p : \mathbb{R}^2 \mapsto \mathbb{R}$, $p(x, y) = x$ is a continuous function (as well as projection $q(x, y) = y$). A sum, difference and product of two continuous functions are continuous functions. A quotient of two continuous function is again a continuous function. A composition of two continuous function is a continuous function.

Thus, $f(x, y) = |x|$ is a continuous function. Indeed, $f(x, y) = |p(x, y)|$ is a composition of p and $|\cdot|$. Thus, $f^{-1}((-\infty, 1)) = \{(x, y) \in \mathbb{R}^2, x \in (-1, 1)\}$ is an open set.

For now: A projection $p : \mathbb{R}^2 \mapsto \mathbb{R}$, $p(x, y) = x$ is a continuous function (as well as projection $q(x, y) = y$). A sum, difference and product of two continuous functions are continuous functions. A quotient of two continuous function is again a continuous function. A composition of two continuous function is a continuous function.

Thus, $f(x, y) = |x|$ is a continuous function. Indeed, $f(x, y) = |p(x, y)|$ is a composition of p and $|\cdot|$. Thus,
$f^{-1}((-\infty, 1)) = \{(x, y) \in \mathbb{R}^2, x \in (-1, 1)\}$ is an open set.
Next, $g(x, y) = y - x^2$ is a continuous function. Indeed,
$g(x, y) = q(x, y) - p(x, y)^2$. Consequently,
$f^{-1}((-\infty, 0)) = \{(x, y) \in \mathbb{R}^2, y - x^2 < 0\} = \{(x, y) \in \mathbb{R}^2, y < x^2\}$.

For now: A projection $p : \mathbb{R}^2 \mapsto \mathbb{R}$, $p(x,y) = x$ is a continuous function (as well as projection $q(x,y) = y$). A sum, difference and product of two continuous functions are continuous functions. A quotient of two continuous function is again a continuous function. A composition of two continuous function is a continuous function.

Thus, $f(x,y) = |x|$ is a continuous function. Indeed, $f(x,y) = |p(x,y)|$ is a composition of p and $|\cdot|$. Thus,
$f^{-1}((-\infty,1)) = \{(x,y) \in \mathbb{R}^2, x \in (-1,1)\}$ is an open set.
Next, $g(x,y) = y - x^2$ is a continuous function. Indeed,
$g(x,y) = q(x,y) - p(x,y)^2$. Consequently,
$f^{-1}((-\infty,0)) = \{(x,y) \in \mathbb{R}^2, y - x^2 < 0\} = \{(x,y) \in \mathbb{R}^2, y < x^2\}$.
Since $M = f^{-1}((-\infty,1)) \cap g^{-1}((-\infty,0))$, we deduce that M is open.

Definition

An *interior* of set $M \subset \mathbb{R}^n$ is a set M^0 of all points x_0 for which there is $r > 0$ such that $B_r(x_0) \subset M$. Equivalently, it is the biggest open set contained in M.
A *closure* of a set $M \subset \mathbb{R}^n$ is a set $\overline{M}$ defined as $\overline{M} := \mathbb{R}^2 \setminus (\mathbb{R}^2 \setminus M)^0$.
Equivalently, it is the smallest closed set containing M.
A *boundary* of a set M is denoted by ∂M and it is defined as $\overline{M} \setminus M^0$.

Definition

An *interior* of set $M \subset \mathbb{R}^n$ is a set M^0 of all points x_0 for which there is $r > 0$ such that $B_r(x_0) \subset M$. Equivalently, it is the biggest open set contained in M.
A *closure* of a set $M \subset \mathbb{R}^n$ is a set $\overline{M}$ defined as $\overline{M} := \mathbb{R}^2 \setminus (\mathbb{R}^2 \setminus M)^0$. Equivalently, it is the smallest closed set containing M.
A *boundary* of a set M is denoted by ∂M and it is defined as $\overline{M} \setminus M^0$.

Example

- Take $M = [0, 1] \times (0, 1)$ and find M^0, $\overline{M}$ and ∂M.

Definition

Let $M \subset \mathbb{R}^n$. A point $x_0 \in \mathbb{R}^2$ is a *limit point* of M if $B_r(x_0) \cap M \neq \emptyset$ for every $r > 0$.
A point $(x_0) \in M$ is an *isolated point* of M if there is $r > 0$ such that $B_r(x_0) \cap M = \{(x_0)\}$.

Definition

Let $M \subset \mathbb{R}^n$. A point $x_0 \in \mathbb{R}^2$ is a *limit point* of M if $B_r(x_0) \cap M \neq \emptyset$ for every $r > 0$.
A point $(x_0) \in M$ is an *isolated point* of M if there is $r > 0$ such that $B_r(x_0) \cap M = \{(x_0)\}$.

Example

- As an example, consider a set $M := \{(x, y) \in \mathbb{R}, y = 0, x = 1/n,\ n \in \mathbb{N}\}$. We claim, that $(0, 0)$ is a limit point of M. Indeed, let $r > 0$. Then there is n_r such that $n_r > 1/r$ and, clearly, $(1/n_r, 0) \in M$ is such point that $\|(1/n_r, 0) - (0, 0)\| < r$ and thus $B_r(0, 0) \cap M = (1/n_r, 0)$.

Definition

We say that $f : M \subset \mathbb{R}^n \mapsto \mathbb{R}$ is continuous at a point $x_0 \in M$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (M \cap B_\delta(x_0)),\ |f(x) - f(x_0)| < \varepsilon.$$

Let $N \subset M$ and let $f : M \mapsto \mathbb{R}$ be continuous at all points $(x_0) \in N$. Then we say that f is continuous on N. If f is continuous on $\mathrm{Dom}\ f$ then we simply say that f is continuous.

Properties of continuous functions

Let f_1 and f_2 be continuous functions. Then

$$f_1 + f_2,\ f_1 - f_2 \text{ and } f_1 f_2$$

are continuous function. Moreover, $\frac{f_1}{f_2}$ whenever it is defined. Further, $f_1 \circ f_2$ is also a continuous function whenever it is correctly defined.

Properties of continuous functions

Let f_1 and f_2 be continuous functions. Then

$$f_1 + f_2,\ f_1 - f_2 \text{ and } f_1 f_2$$

are continuous function. Moreover, $\frac{f_1}{f_2}$ whenever it is defined.
Further, $f_1 \circ f_2$ is also a continuous function whenever it is correctly defined.

Let remind that $f(x, y) = x$ and $f(x, y) = y$ are continuous function.

Properties of continuous functions

Let f_1 and f_2 be continuous functions. Then

$$f_1 + f_2,\ f_1 - f_2 \text{ and } f_1 f_2$$

are continuous function. Moreover, $\frac{f_1}{f_2}$ whenever it is defined. Further, $f_1 \circ f_2$ is also a continuous function whenever it is correctly defined.

Let remind that $f(x, y) = x$ and $f(x, y) = y$ are continuous function.

Example

- Where is

$$f(x, y) = \frac{x + \sqrt{x + y}}{1 + \cos^2 x}$$

continuous?

Definition Let (x_0, y_0) be a limit point of $M \subset \mathbb{R}^2$ and let $f : M \mapsto \mathbb{R}$. We say that a limit of f at the point (x_0, y_0) is $A \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ |f(x, y) - A| < \varepsilon.$$

We write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = A$.

Definition Let (x_0, y_0) be a limit point of $M \subset \mathbb{R}^2$ and let $f : M \mapsto \mathbb{R}$. We say that a limit of f at the point (x_0, y_0) is $A \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ |f(x, y) - A| < \varepsilon.$$

We write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = A$.
We say that a limit of f at the point (x_0, y_0) is ∞ if

$$\forall M > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ f(x, y) > M.$$

We write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \infty$.

Definition Let (x_0, y_0) be a limit point of $M \subset \mathbb{R}^2$ and let $f : M \mapsto \mathbb{R}$. We say that a limit of f at the point (x_0, y_0) is $A \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ |f(x, y) - A| < \varepsilon.$$

We write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = A$.
We say that a limit of f at the point (x_0, y_0) is ∞ if

$$\forall M > 0,\ \exists \delta > 0,\ \forall (x, y) \in (M \cap B_\delta(x_0, y_0)),\ f(x, y) > M.$$

We write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \infty$. We say that a limit of f at the point (x_0, y_0) is $-\infty$ if $\lim_{(x,y)\to(x_0,y_0)} -f(x, y) = -\infty$.

Observation (Arithmetic of limits)

Let f and g be two functions and let (x_0, y_0) be a limit point of $\mathrm{Dom}\ f$ and of $\mathrm{Dom}\ g$. Then

$$\lim_{(x,y)\to(x_0,y_0)} (f+g)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) + \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$

$$\lim_{(x,y)\to(x_0,y_0)} fg(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f}{g}(x,y) = \frac{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}{\lim_{(x,y)\to(x_0,y_0)} g(x,y)}.$$

assuming the right hand side is well defined.

Observation (Arithmetic of limits)

Let f and g be two functions and let (x_0, y_0) be a limit point of $\mathrm{Dom}\ f$ and of $\mathrm{Dom}\ g$. Then

$$\lim_{(x,y)\to(x_0,y_0)} (f+g)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) + \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$

$$\lim_{(x,y)\to(x_0,y_0)} fg(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f}{g}(x,y) = \frac{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}{\lim_{(x,y)\to(x_0,y_0)} g(x,y)}.$$

assuming the right hand side is well defined.

Observation

A function f is continuous at point $(x_0, y_0) \in \mathrm{Dom}\ f$ if and only if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

Example Consider a function

$$f(x, y) = \frac{x^2y^2}{x^2y^2 + (x - y)^2}.$$

This function is not defined at $(0, 0)$. It is possible to define the value $f(0, 0)$ in such a way that f is continuous? In particular, does there exist a finite limit

$$\lim_{(x,y)\to(0,0)} f(x, y)?$$

Sandwich lemma

Let f, g, h be three functions defined on $B_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $\delta > 0$. Assume that for all $(x, y) \neq (x_0, y_0)$ in $B_\delta(x_0, y_0)$ we have

$$g(x, y) \leq f(x, y) \leq h(x, y).$$

If there is $A \in \mathbb{R}^*$ such that

$$\lim_{(x,y)\to(x_0,y_0)} g(x, y) = \lim_{(x,y)\to(x_0,y_0)} h(x, y) = A$$

then also

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = A.$$

Figure: Sandwich lemma

Examples

- $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$
- $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x-2y}{3x+y}\right)$
- $\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x-2y}{3x+y}\right)$
- $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4}$
- $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

Derivatives

Derivatives

Definition

Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$ be such that $\|v\| = 1$. Let $x_0 \in M^0$. The derivative of f with respect to direction v in a point x_0 is

$$Df(x_0, v) = g'(t)|_{t=0} \text{ where } g(t) = f(x_0 + tv)$$

Derivatives

Definition

Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$ be such that $\|v\| = 1$. Let $x_0 \in M^0$. The derivative of f with respect to direction v in a point x_0 is

$$Df(x_0, v) = g'(t)|_{t=0} \text{ where } g(t) = f(x_0 + tv)$$

The direction of an arbitrary vector v is a unit vector $\frac{v}{\|v\|}$.

Derivatives

Definition

Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$ be such that $\|v\| = 1$. Let $x_0 \in M^0$. The derivative of f with respect to direction v in a point x_0 is

$$Df(x_0, v) = g'(t)|_{t=0} \text{ where } g(t) = f(x_0 + tv)$$

The direction of an arbitrary vector v is a unit vector $\frac{v}{\|v\|}$.

Examples

- What is the direction of a line $p : (x, y) = (2, -1) + t(1, 3)$?
- Let $f(x, y) = x^2 e^y$. Compute $Df\left((1, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right)$.

Partial derivatives

Definition

Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ and let $\overline{x} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n) \in M^0$. Let e_i be a vector in direction of the axis x_i.

$$\frac{\partial}{\partial x_i} f(\overline{x}) = Df((\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n), e_i),$$

is called a **partial derivative** with respect to x_i in point x.

Partial derivatives

Definition

Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ and let $\overline{x} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n) \in M^0$. Let e_i be a vector in direction of the axis x_i.

$$\frac{\partial}{\partial x_i} f(\overline{x}) = Df((\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n), e_i),$$

is called a **partial derivative** with respect to x_i in point x.

Example

- Let $f(x,y) = \frac{x(\sin y)}{1+x^2}$. Compute $\frac{\partial}{\partial x} f((1,1))$ and $\frac{\partial}{\partial y} f((1,1))$.
- Compute $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ for the function from the previous exercise.

Partial derivatives

Definition

Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ and let $\overline{x} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n) \in M^0$. Let e_i be a vector in direction of the axis x_i.

$$\frac{\partial}{\partial x_i} f(\overline{x}) = Df((\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n), e_i),$$

is called a **partial derivative** with respect to x_i in point x.

Example

- Let $f(x, y) = \frac{x(\sin y)}{1+x^2}$. Compute $\frac{\partial}{\partial x} f((1,1))$ and $\frac{\partial}{\partial y} f((1,1))$.
- Compute $\frac{\partial}{\partial x} f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ for the function from the previous exercise.

Remark: the gradient of f is a vector of its partial derivative. Namely

$$\nabla f = \left(\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \dots, \frac{\partial}{\partial x_n} f \right).$$

Definition

We define the second order derivatives as follows

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right), \; \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

whenever $i, j \in \{1, \ldots, n\}$, $i \neq j$. Analogously, we define the third and higher order derivatives

Example

- Compute all second order derivatives of

$$f(x, y) = \frac{x}{y} - e^{xy}.$$

Definition

We define the second order derivatives as follows

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right), \ \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

whenever $i, j \in \{1, \ldots, n\}$, $i \neq j$. Analogously, we define the third and higher order derivatives

Example

- Compute all second order derivatives of

$$f(x, y) = \frac{x}{y} - e^{xy}.$$

Let $f \in C^2$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

In particular, $\nabla^2 f$ is a symmetric n by n matrix.

Theorem (Chain rule)

Let $n, m \in \mathbb{N}$, *let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *and* $g : \mathbb{R}^m \to \mathbb{R}$. *Then*

$$\frac{\partial (g \circ f)}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial g}{\partial y_j} \frac{\partial f_j}{\partial x_i}$$

for every $i \in \{1, \ldots, n\}$.

Theorem (Chain rule)

Let $n, m \in \mathbb{N}$, let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}$. Then

$$\frac{\partial (g \circ f)}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial g}{\partial y_j} \frac{\partial f_j}{\partial x_i}$$

for every $i \in \{1, \ldots, n\}$.

Example

- Let $f(x) = g(\sin x, \cos x)$ (we use notation $g = g(a, b)$). Then

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial a} \cos x - \frac{\partial g}{\partial b} \sin x.$$

- Calculate $\frac{\partial f}{\partial t}$ where
  1. $f(x, y) = 4x^2 + 3y^2$, $x = x(t) = \sin t$ and $y = y(t) = \cos t$,
  2. $f(x, y) = \sqrt{x^2 - y^2}$, $x = x(t) = e^{2t}$ and $y = y(t) = e^{-t}$.

Differential

Consider a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$. We try to compute an increment of a function if we move from the point (x_0, y_0) to the point $(x_0 + h, y_0 + k)$, i.e., $\Delta f(x_0, y_0) = f(x_0 + h, y_0 + k) - f(x_0, y_0)$. It can be written as

$$\Delta f(x_0, y_0) = f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) + f(x_0 + h, y_0) - f(x_0, y_0).$$

Assuming $|h|$ and $|k|$ are sufficiently small we can us an approximation

$$f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) \sim \frac{\partial f}{\partial x}(x_0 + h, y_0)k$$
$$f(x_0 + h, y_0) - f(x_0, y_0) \sim \frac{\partial f}{\partial y}(x_0, y_0)h$$

Moreover, $\frac{\partial f}{\partial x}(x_0 + h, y_0) \sim \frac{\partial f}{\partial x}(x_0, y_0)$ if $f \in C^1$. This yields

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) \sim \frac{\partial f}{\partial x}(x_0, y_0)h + \frac{\partial f}{\partial y}(x_0, y_0)k.$$

We denote by $\mathrm{d}x$ the change in the x coordinate and $\mathrm{d}y$ the change in the y coordinate.

Definition

Let $f : M \subset \mathbb{R}^2 \to \mathbb{R}$ be such that f has continuous first partial derivatives. Then

$$\mathrm{d}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\mathrm{d}x + \frac{\partial f}{\partial y}(x_0, y_0)\mathrm{d}y$$

is called the *differential* of f at the point (x_0, y_0).

We denote by $\mathrm{d}x$ the change in the x coordinate and $\mathrm{d}y$ the change in the y coordinate.

Definition

Let $f : M \subset \mathbb{R}^2 \to \mathbb{R}$ be such that f has continuous first partial derivatives. Then

$$\mathrm{d}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\mathrm{d}x + \frac{\partial f}{\partial y}(x_0, y_0)\mathrm{d}y$$

is called the *differential* of f at the point (x_0, y_0).

Example

- Determine an approximate value of $\sqrt{(0.03)^2 + (2.89)^2}$ by use of the differential.

We denote by $\mathrm{d}x$ the change in the x coordinate and $\mathrm{d}y$ the change in the y coordinate.

Definition

Let $f : M \subset \mathbb{R}^2 \to \mathbb{R}$ be such that f has continuous first partial derivatives. Then

$$\mathrm{d}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\mathrm{d}x + \frac{\partial f}{\partial y}(x_0, y_0)\mathrm{d}y$$

is called the *differential* of f at the point (x_0, y_0).

Example

- Determine an approximate value of $\sqrt{(0.03)^2 + (2.89)^2}$ by use of the differential.

It is worth to mention that $\mathrm{d}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot (\mathrm{d}x, \mathrm{d}y)$. In multiple dimension,

$$\mathrm{d}f(x_0) = \nabla f(x_0) \cdot (\mathrm{d}x_1, \dots, \mathrm{d}x_n).$$

Definition

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ have continuous partial derivatives at point $x_0 \in \mathbb{R}^n$. Then a *tangent plane* of the graph of f at point x_0 is a plane with equation

$$z = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

Definition

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ have continuous partial derivatives at point $x_0 \in \mathbb{R}^n$. Then a *tangent plane* of the graph of f at point x_0 is a plane with equation

$$z = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

Example

- Compute a tangent plane of the graph of f at point $(1, 2)$ for $f(x, y) = \sqrt{9 - x^2 - y^2}$.

Definition

We define the second order Taylor polynomial at a point $x_0 \in \mathbb{R}^n$ as

$$T_2(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)\nabla^2 f(x - x_0)^T$$

Definition

We define the second order Taylor polynomial at a point $x_0 \in \mathbb{R}^n$ as

$$T_2(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)\nabla^2 f(x - x_0)^T$$

Example

- Find an approximate value of

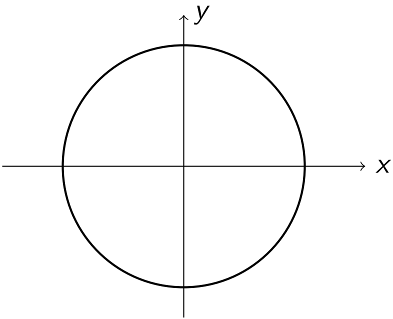
$$\sqrt{(0.03)^2 + (2.89)^2}$$

by use of the second order Taylor polynomial

Implicitly given function

First, consider a set

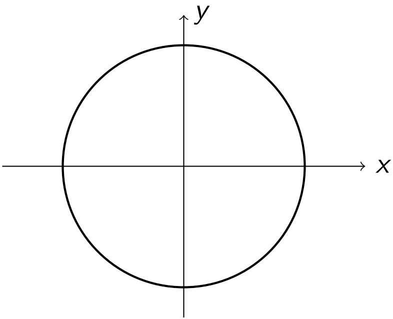
$$\{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$$



Implicitly given function

First, consider a set

$$\{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$$



The equation $x^2 + y^2 = 1$ define two function $y_1(x)$ and $y_2(x)$ where

$$y_1(x) = \sqrt{1 - x^2},\ \mathrm{Dom}\ y_1(x) = [-1, 1],$$
$$y_2(x) = -\sqrt{1 - x^2},\ \mathrm{Dom}\ y_2(x) = [-1, 1].$$

What if it is impossible to express y? Consider an equation

$$f(x,y) = 0.$$

What assumptions should be imposed in order to get uniquely defined function $y(x)$?

What if it is impossible to express y? Consider an equation

$$f(x,y)=0.$$

What assumptions should be imposed in order to get uniquely defined function $y(x)$?

Theorem

Let $f:\mathbb{R}^2\mapsto\mathbb{R}$ and $(x_0,y_0)\in\mathbb{R}^2$ be given. If

- i $f\in C^k$ *for some* $k\in\mathbb{N}$,
- ii $f(x_0,y_0)=0$,
- iii $\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$,

Then there is a uniquely determined function $y(x)$ of class C^k on a neighborhood of point x_0 such that $f(x,y(x))=0$ (precisely, there is $\epsilon>0$ and a function $y(x)$ defined on $(x_0-\varepsilon,x_0+\varepsilon)$ such that $f(x,y(x))=0$.

Example Consider an equation

$$x^3 + y^3 - 3xy - 3 = 0.$$

Is there a function $y(x)$ determined by the given equation on the neighborhood of a point $(1, 2)$?

Note that the last assumption in the implicit function theorem cannot be omited. Consider the first equation

$$x^2 + y^2 = 1$$

and let decide whether there is a function $y(x)$ given by that equation at the point $(1, 0)$. According to the picture, it is impossible (recall the vertical line test). The theorem may not be applied. Take $f(x, y) = x^2 + y^2 - 1$. We have

$$\frac{\partial f}{\partial y} = 2y, \ \frac{\partial f}{\partial y}(1, 0) = 0$$

and the third assumption is not fulfilled.

Or another example, consider a set

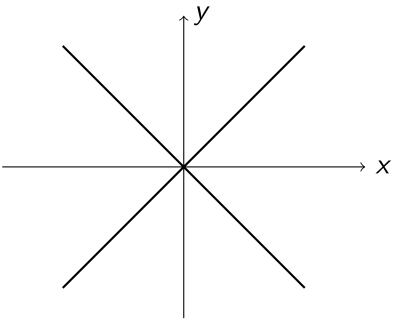
$$\{(x, y) \in \mathbb{R}^2,\ x^2 - y^2 = 0\}.$$

Is this set a graph of some function around a point $(0, 0)$?

Or another example, consider a set

$$\{(x, y) \in \mathbb{R}^2,\ x^2 - y^2 = 0\}.$$

Is this set a graph of some function around a point $(0, 0)$?



Further analysis of the implicitly given function

In order to examine further qualitative properties of the given function we have to compute derivatives at the given points. The easiest method is to differentiate the given equation with respect to x (and to assume that y is in fact a function of x).

Example

Consider an equation

$$e^{2x} + e^{y} + x + 2y - 2 = 0.$$

Does this equation define a function $y(x)$ on a neighborhood of $(0, 0)$. If yes, compute $y'(0)$ and $y''(0)$.

Some relations

What is the relation for $y'(x_0)$? And what is the relation for $y''(x_0)$?

Some relations

What is the relation for $y'(x_0)$? And what is the relation for $y''(x_0)$?

$$y'(x_0) = -\frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)}.$$

Some relations

What is the relation for $y'(x_0)$? And what is the relation for $y''(x_0)$?

$$y'(x_0) = -\frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)}.$$

Example

- Show that there is a function $y(x)$ given by an equation $y - \frac{1}{2}\sin y = x$ on a neighborhood of (π, π). Find the tangent line to $y(x)$ at the point $x_0 = \pi$.

Extremes, local extremes

Definition

Let $f : M \subset \mathbb{R}^n \mapsto \mathbb{R}$. We say that f attains a *local maximum* at a point $x_0 \in M^0$ if there is $r > 0$ such that $f(x_0) \geq f(x)$ for all $(x) \in B_r(x_0)$. We say that f attains a *local minimum* at a point $x_0 \in M^0$ if there is $r > 0$ such that $f(x_0) \leq f(x)$ for all $(x) \in B_r(x_0)$.

Extremes, local extremes

Definition

Let $f : M \subset \mathbb{R}^n \mapsto \mathbb{R}$. We say that f attains a *local maximum* at a point $x_0 \in M^0$ if there is $r > 0$ such that $f(x_0) \geq f(x)$ for all $(x) \in B_r(x_0)$. We say that f attains a *local minimum* at a point $x_0 \in M^0$ if there is $r > 0$ such that $f(x_0) \leq f(x)$ for all $(x) \in B_r(x_0)$.

How to examine the extremes? Recall the Taylor polynomial of the second order

$$T_2(x_0)(f)(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)(\nabla^2 f(x_0))(x - x_0)^T.$$

Lemma

Let $f \in C^1$ have a local extreme at x_0. Then $\nabla f(x_0) = 0$.

Lemma

Let $f \in C^1$ *have a local extreme at* x_0. *Then* $\nabla f(x_0) = 0$.

Definition

A point $(x_0) \in M$ such that $\nabla f(x_0) = 0$ is called a **stationary point**.

Lemma

Let $f \in C^1$ have a local extreme at x_0. Then $\nabla f(x_0) = 0$.

Definition

A point $(x_0) \in M$ such that $\nabla f(x_0) = 0$ is called a **stationary point**.

Lemma

Let $f \in C^2$ and let x_0 be its stationary point. Then

- *if $\nabla^2 f(x_0)$ is positive-definite, then f has a local minimum at x_0,*
- *if $\nabla^2 f(x_0)$ is negative-definite, then f has a local maximum at x_0,*
- *if $\nabla^2 f(x_0)$ is indefinite, then there is no extreme at x_0,*
- *otherwise, we do not know anything.*

Example

- Examine local extremes of

$$f(x,y) = x^3 + 3xy^2 - 15x - 12y.$$

Global extremes with respect to a set

Lemma

Let $M \subset \mathbb{R}^n$ and let $f : M \to \mathbb{R}$. Then f attains its **minimum on** *M at point $(x_0, y_0) \in \mathbb{M}$ if*

$$\forall (x, y) \in M, \; f(x_0, y_0) \leq f(x, y).$$

Similarly, f attains its **maximum on** *M at point $(x_0, y_0) \in \mathbb{M}$ if*

$$\forall (x, y) \in M, \; f(x_0, y_0) \geq f(x, y).$$

Global extremes with respect to a set

Lemma

Let $M \subset \mathbb{R}^n$ and let $f : M \to \mathbb{R}$. Then f attains its **minimum on** *M at point $(x_0, y_0) \in \mathbb{M}$ if*

$$\forall (x, y) \in M,\ \ f(x_0, y_0) \leq f(x, y).$$

Similarly, f attains its **maximum on** *M at point $(x_0, y_0) \in \mathbb{M}$ if*

$$\forall (x, y) \in M,\ \ f(x_0, y_0) \geq f(x, y).$$

Lemma

Let $M \subset \mathbb{R}^n$ be a bounded and closed set and let $f : M \to \mathbb{R}$ be a continuous function. Then f attains its minimum and maximum on M.

Example

- Find the maximum and minimum of

$$f(x, y) = (x^2 + y)e^y$$

on a set

$$M = \{(x, y) \in \mathbb{R}^2,\ y \geq \frac{1}{3}x,\ y \leq 3x,\ y \leq 5 - x\}.$$

- Give two examples of functions (and sets M) which do not attain their extremes.

Reminder: Exercises

- Find all local maxima and minima of

$$f(x,y) = 3y^3 - x^2y^2 + 8y^2 + 4x^2 - 20y.$$

- Find the points where the function

$$f(x,y) = x^2 + y^2 - xy - x - 2$$

considered on a rectangle

$$M = \{(x,y) \in \mathbb{R}^2,\, 0 \leq x \leq 2,\, 0 \leq y \leq 1\}$$

attains its maximum and minimum.

- Find the maximum and minimum values of

$$f(x,y) = 81x^2 + y^2$$

subject to the constraint $4x^2 + y^2 \leq 9$.

Theorem (The Lagrange multipliers)

Let $f : \mathrm{Dom}\, f \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^1 *function defined on a neighborhood of*

$$M = \{x \in \mathbb{R}^n, g(x) = 0\}$$

where g *is a* C^1 *function. If there is an extreme of* f *with respect to the set* M *then there exists* $\lambda \in \mathbb{R}$ *such that*

$$\nabla f + \lambda \nabla g = 0.$$

Exercises

- Find the maximum and minimum values of

$$f(x, y, z) = y^2 - 10z$$

subject to the constraint

$$x^2 + y^2 + z^2 = 36.$$

- Find extremes of

$$f(x, y) = x^2 + y^2 - 12x - 16y$$

on

$$M = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \leq 25,\ x \geq 0\}.$$

Theorem (The Lagrange multipliers - two constraints)

Let $n \geq 3$, $f : \mathrm{Dom}\ f \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^1 *function defined on a neighborhood of*

$$M = \{x \in \mathbb{R}^n,\ g(x) = 0,\ \ h(x) = 0\}$$

where $g, h : \mathbb{R}^n \to \mathbb{R}$ *are* C^1 *functions. If there is an extreme of* f *with respect to the set* M *then there exists* $\lambda,\ \mu \in \mathbb{R}$ *such that*

$$\nabla f + \lambda \nabla g + \mu \nabla h = 0.$$

Theorem (The Lagrange multipliers - two constraints)

Let $n \geq 3$, $f : \mathrm{Dom}\, f \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function defined on a neighborhood of

$$M = \{x \in \mathbb{R}^n, \ g(x) = 0, \ h(x) = 0\}$$

where $g, h : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions. If there is an extreme of f with respect to the set M then there exists $\lambda, \mu \in \mathbb{R}$ such that

$$\nabla f + \lambda \nabla g + \mu \nabla h = 0.$$

Example

- Find the maximum and minimum values of

$$f(x, y, z) = 3x^2 + y$$

subject to the constraints

$$4x - 3y = 9 \quad \text{and } x^2 + z^2 = 9.$$

Applications to economics

Applications to economics

- Suppose you are running a factory producing some sort of widget that requires steel as a raw material. Your costs are predominantly human labor, which is \$20 per hour for your worker, and the steel itself, which runs for \$170 per ton. Suppose your revenue R is loosely modeled by the following equation

$$R(h, s) = 200h^{2/3}s^{1/3}$$

where h represents hours of labor and s represents tons of steel. If your budget is \$20 000, what is the maximum possible revenue?

- The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize the cost.

- A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dolars and the deluxe at y dollars, then the manufacturer will sell $500(y - x)$ of the standard items and $45\,000 + 500(x - 2y)$ of the deluxe each year. How should the items be priced to maximize the profit?

- Assume that the cost of a car (of one given type) depends linearly on its age, i.e.,

$$y = ax + b, \ \ a, b \in \mathbb{R},$$

where y is the price of a car and x is its age. Determine the dependence from the following data using the **least square method**.

x	2	3	3	3	4	4	5	5	6
y	28.7	24.8	26.0	30.5	23.8	24.6	23.8	20.4	22.1