# Linear Algebra

Václav Mácha

University of Chemistry and Technology

# Introduction to linear algebra

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Why linear algebra?

## Definition

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- sum, i.e. $\forall v_1, v_2 \in V,\ v_1 + v_2 \in V$,
- multiplication by real number, i.e. $\forall \alpha \in \mathbb{R},\ \forall v \in V,\ \alpha v \in V$.

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We assume that the set $V$ is closed with respect to both operations (i.e., all possible results belong to $V$). Moreover,

- the summation is associative, commutative, there is $0 \in V$ and for all $v \in V$ there is $-v$,
- the multiplication satisfies $\alpha(\beta v_1) = (\alpha\beta)v_1$, $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$, $(\alpha + \beta)v_1 = \alpha v_1 + \beta v_1$, $1v_1 = v_1$ for all $\alpha, \beta \in \mathbb{R}$ and $v_1, v_2 \in V$.

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Elements of $V$ are called **vectors**.

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- the space of ordered pairs of real numbers $(u, v) \in \mathbb{R}^2$ with summation and product defined as

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \quad \alpha(u_1, v_1) = (\alpha u_1, \alpha v_1)$$

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for all $(u_1, v_1),\ (u_2, v_2) \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$.
- the space of all $(x, y) \in \mathbb{R}^2$ satisfying the equation

$$x + 2y = 0. \tag{1}$$

also forms a vector space with summation and product defined as above.

- on the other hand, the space of all $(x, y) \in \mathbb{R}^2$ satisfying

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- The set of all polynomials (with usually defined summation and product) is a vector space.
- The set of second degree polynomials is not a vector space.
- The set of polynomials of degree 0, 1 and two form a vector space.

## Definition

Let $S \subset V$ be such that for all $s_1, s_2 \in S$ and $\alpha \in \mathbb{R}$ it holds that $s_1 + s_2 \in S$ and $\alpha s_1 \in S$. Then $S$ itself is a vector space and we say that $S$ is a **subspace** of $V$. If $S$ is nonempty and $S \neq V$ then we say $S$ is a **proper subspace**.

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Examples:

- a subset $S = \{(x, y, 0) \in \mathbb{R}^3\}$ of $V = \{(x, y, z) \in \mathbb{R}^3\}$.
- (from the previous slide) all $(x, y) \in \mathbb{R}^2$ solving $x + 2y = 0$.

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Examples (and an exercise):

- $(2, 5, 3)$ is a linear combination of $(1, 1, 0)$ and $(0, 1, 1)$ because

$$(2, 5, 3) = 2(1, 1, 0) + 3(0, 1, 1).$$

- $(0, 2, -1)$ is not a linear combination of $(1, 1, 0)$ and $(0, 1, 1)$.
- Is $(1, 1, 0, 5)$ a linear combination of $(1, -1, 0, 0)$, $(2, 0, 2, 1)$ and $(0, -1, 0, 2)$?

## Definition

The set of all linear combinations of $v_1, v_2, \ldots, v_n$ is called a **linear span** of a set $\{v_1, v_2, \ldots, v_n\}$. We denote it by $\mathrm{span}\{v_1, v_2, \ldots, v_n\}$.

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Mathematically speaking:

$$\operatorname{span}\{v_1, v_2, \ldots, v_n\} = \left\{ \sum_{i=1}^{n} \alpha_i v_i,\ \alpha_i \in \mathbb{R}\ \forall i \in \{1, \ldots, n\} \right\}.$$

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## Lemma

*Linear span is a vector space*

Example:

- The set $\{(x, y, z) \in \mathbb{R}^3, 2x + y - z = 0\}$ contains a span of $v_1 = (1, -2, 0)$ and $v_2 = (0, 1, 1)$ (or, for example, $w_1 = (1, 0, 2)$ and $w_2 = (1, 1, 3)$).

## Definition

Vectors $v_1, v_2, \ldots, v_n$ are said to be **linearly dependent** if the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$$

has a nontrivial solution (i.e. a solution $\alpha_1, \alpha_2, \ldots, \alpha_n$ where at least one coefficient is nonzero).

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has only solution $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Examples:

- Vectors $(1, 0)$, $(0, 1)$ and $(-2, 3)$ are linearly dependent.
- Vectors $(1, 1, 0)$, $(2, 2, 0)$ and $(-1, 0, 1)$ are linearly dependent.
- Vectors $(2, 3, 1, 0)$, $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$ are linearly independent.

## Lemma

*Let $v_1, v_2, \ldots, v_n$ be linearly dependent. Then one of the vectors is a linear combination of the remaining vectors. Precisely, there is $i \in \{1, \ldots, n\}$ such that $v_i \in span\{\{v_1, v_2, \ldots, v_n\} \setminus \{v_i\}\}$.*

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## Proof.

According to assumption, there is $i \in \{1, \dots, n\}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

has a solution with $\alpha_i \neq 0$. Assume, without lost of generality, that $i = 1$. We may rearrange the equation to a form

$$v_1 = -\frac{\alpha_2}{\alpha_1} v_2 - \frac{\alpha_3}{\alpha_1} v_3 - \dots - \frac{\alpha_n}{\alpha_1} v_n.$$

$\square$

## Lemma

*Let* $v_1 \in span\{v_2, \dots, v_n\}$. *Then*

$$span\{v_2, \dots, v_n\} = span\{v_1, v_2, \dots, v_n\}.$$

## Proof.

Clearly, $\text{span}\{v_2, \dots, v_n\} \subset \text{span}\{v_1, v_2, \dots, v_n\}$. Next, let

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

Since $v_1 = \sum_{i=2}^{n} \beta_i v_i$ for some $\beta_i \in \mathbb{R}$, we get

$$v = \sum_{i=2}^{n} (\alpha_i + \alpha_1 \beta_i) v_i$$

and $v \in \text{span}\{v_2, \dots, v_n\}$. $\square$

## Definition

Let $V = \text{span}\{v_1, v_2, \ldots, v_n\}$. Then we say that $\{v_1, v_2, \ldots, v_n\}$ **generates** $V$. The vectors $\{v_1, v_2, \ldots, v_n\}$ are called **generators** of $V$.

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### Definition

We say, that $V$ is of **dimension** $n \in \mathbb{N}$ iff every basis has $n$ elements.

Examples:

- The set $\{(1,0),(0,1)\} \subset \mathbb{R}^2$ generates $\mathbb{R}^2$. Moreover, these vectors are linearly independent. Consequently, the dimension of $\mathbb{R}^2$ is 2.
- Vectors $\{1, x, x^2\}$ generates the space of polynomials of degree at most 2 and they are linearly independent.

Let $\{v_i, i = 1, \ldots, n\}$ be a set of independent vectors. Then it forms basis of $\text{span}\{v_i, i = 1, \ldots, n\}$ and the dimension of that linear span is $n$.

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Let $\{v_i, i = 1, \ldots, n\}$ be independent vectors and let $v \in \text{span}\{v_i, i = 1, \ldots, n\}$. Then the numbers $\alpha_i,\ i = 1, \ldots, n$ satisfying $v = \sum_{i=1}^{n} \alpha_i v_i$ are determined uniquely and they are called **coordinates** of $v$ with respect to the given basis.

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**Examples**

- Find the coordinates of $u = (0, 1)$ with respect to a basis $\{v, w\}$ where $v = (3, 2)$, and $w = (4, 3)$.
- Find the coordinates of $x = (-1, 0, 2)$ with respect to a basis $(u, v, w)$ where $u = (2, 1, 1)$, $v = (1, -1, 1)$, and $w = (1, 1, 1)$

## Matrices, introduction

### Definition

A **matrix** is a table of numbers arranged in rows and columns. Namely, let $m, n$ be natural numbers. Then

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})_{i=1,j=1}^{m,n}$$

The matrix $A$ has $m-$rows and $n-$columns. The matrix $A$ is said to be of type $(m, n)$.

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For example matrix

$$\begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & -1 \end{pmatrix}$$

has two rows and three columns and it is of type $(2, 3)$ (or it is of type two by three).

## Operations with matrices

**summation**: Let $A = (a_{ij})_{i=1,j=1}^{m,n}$ and $B = (b_{ij})_{i=1,j=1}^{m,n}$ be two matrices of the same type. Then we define

$$A + B = (a_{ij} + b_{ij})_{i=1,j=1}^{m,n}.$$

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Example:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 & -5 \\ 1 & 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 & -5 \\ 1 & 1 & -2 & 2 \end{pmatrix}.$$

!!! Matrices of different types cannot be summed !!!

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**multiplication by real number**: Let $\alpha \in \mathbb{R}$. Then $\alpha A = (\alpha a_{ij})_{i=1,j=1}^{m,n}$.

Example:

$$3 \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & \frac{3}{2} \\ 6 & 6 \\ -9 & 3 \end{pmatrix}$$

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Matrices of given type $(m, n)$ forms a vector space.

**transposition**: For a matrix $A = (a_{ij})_{i=1,j=1}^{m,n}$ we define a transpose matrix $A^T$ as

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Examples:

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 3 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 & -1 & -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 3 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

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Another example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Definition**

If $A = A^T$, $A$ is called a **symmetric** matrix.

## Matrix multiplication

Let $A$ be of type $(m, n)$ and $B$ be of type $(n, p)$. Then $C := AB$ of type $(m, p)$ is defined as

$$C = (c_{ij})_{i=1,j=1}^{m,p}$$

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$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

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Example:

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**Warning:**

$$AB \neq BA$$

Example: A vendor sells hot dogs and corn dogs at three different locations. His total sales(in hundreds) for January and February from the three locations are given in the table below.

| | January h.d | January c.d. | February h.d. | February c.d. |
|---|---|---|---|---|
| place 1 | 10 | 8 | 8 | 7 |
| place 2 | 8 | 6 | 6 | 7 |
| place 3 | 6 | 4 | 6 | 5 |

- Represent this tables as 3 times 2 matrices $J$ and $F$.
- Determine the total sales for the two months, that is $J + F$.
- If hot dogs sell for 3 dollars and corn dogs for 2 dollars, find the revenue from the sale of hot dogs and corn dogs.

## Definition

A **rank of matrix** $A$ is a dimension of vector space generated by its rows. It is denoted by $\mathrm{rank} A$.

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## Lemma

*It holds that* $rankA = rankA^T$.

### Examples

- Determine a rank of

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

- Determine a rank of

$$\begin{pmatrix} 1 & 8 & 1 \\ 1 & 2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

## Systems of equations

We are going to deal with system of $m$ linear equations with $n$ unknowns $x_1,\ x_2, \dots,\ x_n$.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}$$

## Systems of equations

We are going to deal with system of $m$ linear equations with $n$ unknowns $x_1,\ x_2, \dots,\ x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

We use notation $x = (x_1, x_2, \dots, x_n)$, $b = (b_1, b_2, \dots, b_n)$ and $A = (a_{ij})_{i=1,j=1}^{mn}$. Then the above system may be rewritten as

$$Ax^T = b^T.$$

## Definition

An **elementary transformation** is

- scaling the entire row with a nonzero real number or
- interchanging the rows within a matrix or
- adding $\alpha-$multiple of one row to another (here $\alpha \in \mathbb{R}$).

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Let $A$ arise from $B$ by one or more elementary transformations. Then we write $A \sim B$.

For example

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 4 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 4 \\ 6 & -7 \end{pmatrix}.$$

## Definition

A **leading coefficient** of a row is the first non-zero coefficient in that row. We say that matrix $A$ is in **echelon form** if the leading coefficient (also called a **pivot**) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

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Examples: Consider following matrices

$$A = \begin{pmatrix} -1 & -1 & 3 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & -1 & 3 & 0 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

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The matrix $A$ is in echelon form whereas the matrix $B$ is not in echelon form.

## Lemma

*Let $A$ be in echelon form. Then its rank is equal to the number of non-zero rows.*

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*Let $A \sim B$. Then $rank(A) = rank(B)$.*

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Examples:

- The rank of

$$A = \begin{pmatrix} 2 & -1 & 1 & 3 \\ 3 & 2 & 2 & 1 \\ 1 & 3 & 1 & -2 \end{pmatrix}.$$

  is 2

## Lemma

*Let $A \sim B$. Then $rank(A) = rank(B)$.*

Examples:

- The rank of
$$A = \begin{pmatrix} 2 & -1 & 1 & 3 \\ 3 & 2 & 2 & 1 \\ 1 & 3 & 1 & -2 \end{pmatrix}.$$
is 2
- Vectors $(1, 0, 1)$, $(0, 1, 0)$, $(-1, 0, -1)$, $(1, 1, 2)$ are linearly dependent.
- Vectors $(1, 2, 2, -1)$, $(3, 1, 0, 1)$ and $(-1, 3, 4, -3)$ are linearly dependent.
- Vectors $(2, 1, 1)$, $(1, 1, 0)$ and $(0, 1, 0)$ are linearly independent.

The system of equations will be represented by an augmented matrix – i.e. a matrix $A = (a_{i,j})_{i=1,j=1}^{mn}$ with extra column of the right hand side. For example, a system of equations

$$\begin{aligned} 2x + 5y &= 10 \\ 3x + 4y &= 24 \end{aligned}$$

is represented by an augmented matrix

$$\left(\begin{array}{cc|c} 2 & 5 & 10 \\ 3 & 4 & 24 \end{array}\right).$$

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is represented by an augmented matrix

$$\begin{pmatrix} 2 & 5 & | & 10 \\ 3 & 4 & | & 24 \end{pmatrix}.$$

Such matrix consists of two parts – matrix $A = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$ and a vector of right hand side $b = (3, 4)$. Then the augmented matrix can be written as $\left(A|b^T\right)$.

Let solve the system from the previous slide by Gauss elimination:

$$\left(\begin{array}{cc|c} 2 & 5 & 10 \\ 3 & 4 & 24 \end{array}\right) \sim \left(\begin{array}{cc|c} 6 & 15 & 30 \\ 3 & 4 & 24 \end{array}\right) \sim \left(\begin{array}{cc|c} 6 & 15 & 30 \\ 6 & 8 & 48 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 6 & 15 & 30 \\ 0 & -7 & 18 \end{array}\right)$$

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The last row of the last matrix represent an equation

$$-7y = 18 \;\Rightarrow\; y = -\frac{18}{7}.$$

The first row of the last matrix represent

$$6x + 15y = 30$$

and once we plug there $y = -\frac{18}{7}$ we deduce

$$x = \frac{80}{7}.$$

## Theorem (Frobenius)

*A system of linear equations has a solution if and only if* $rank A = rank(A|b^T)$.

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$$\begin{aligned} -x + y + z &= 0 \\ 2y + x + z &= 1 \\ 2z + 3y &= 2. \end{aligned}$$

## Theorem (Frobenius)

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Exercise: Solve

$$\begin{aligned} -x+y+z&=0\\ 2y+x+z&=1\\ 2z+3y&=2. \end{aligned}$$

We have

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0\\ 1 & 2 & 1 & 1\\ 0 & 3 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0\\ 0 & 3 & 2 & 1\\ 0 & 3 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0\\ 0 & 3 & 2 & 1\\ 0 & 0 & 0 & 1 \end{array}\right)$$

and, according to the Frobenius theorem, there is no solution to the given system.

## Non-unique solutions

Solve

$$2x + y - z = 3$$
$$x - 2y + 3z = -1$$

## Non-unique solutions

Solve

$$\begin{aligned} 2x + y - z &= 3 \\ x - 2y + 3z &= -1 \end{aligned}$$

Gauss elimination:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & -2 & 3 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 2 & 1 & -1 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 5 & -7 & 5 \end{array}\right)$$

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$x$, $y$ dependent variables, $z$ free variable.

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$x$, $y$ dependent variables, $z$ free variable.

Solutions:

$$(x, y, z) = (1, 1, 0) + t\left(-\frac{1}{5}, \frac{7}{5}, 1\right).$$

The last exercise: Solve

$$\begin{aligned} -x + py + pz &= 1 \\ x + y + pz &= 2 \\ px + y + 2pz &= 5 - 2x \end{aligned}$$

where $p$ is a real parameter.

# Square matrices

## Definition

A matrix $I$ of type $(n, n)$ is called an **identity matrix** if $I = (a_{ij})_{i=1,j=1}^{nn}$, $a_{ii} = 1$ for all $i \in \{1, \ldots, n\}$ and $a_{ij} = 0$ whenever $i \neq j$.

For example,

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $n = 3$. It holds that $AI = IA = A$ for every matrix $A$ of type $(n, n)$.

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## Definition

Let $A$ by a matrix of type $(n, n)$. If there is a matrix $B$ of type $(n, n)$ such that

$$AB = BA = I$$

then $B$ will be called an **inverse matrix** to $A$ and we use notation $B = A^{-1}$. If there is $A^{-1}$, $A$ is called a **regular matrix**, otherwise it is a **singular matrix**.

Computations:

Verify that

$$\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}.$$

Use it to compute the unknown matrix $X$:

$$\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} X + \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -2 & 3 \end{pmatrix}$$

and the unknown matrix $Y$:

$$Y \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 3 \end{pmatrix}.$$

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and the unknown matrix $Y$:

$$Y \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 3 \end{pmatrix}.$$

Solve

$$\begin{aligned} 2x - y &= 6 \\ -5x + 3y &= 2. \end{aligned}$$

**Lemma**

*Let $A$ be a regular matrix. Then a system $Ax^T = b^T$ has a unique solution.*

**Proof**: It suffices to apply $A^{-1}$ from the left hand side on both sides of the equation. One gets

$$x^T = A^{-1}b^T.$$

The Gauss elimination may be used to find $A^{-1}$. In particular, one has to write down an augmented matrix $(A|I)$ and use elementary transformations to get $(I|B)$. If this is possible, then $B = A^{-1}$.

Exercise: find $A^{-1}$ to $A = \begin{pmatrix} 2 & -1 \\ 3 & -3 \end{pmatrix}$:

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Exercise: find $A^{-1}$ to $A = \begin{pmatrix} 2 & -1 \\ 3 & -3 \end{pmatrix}$:

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & -2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 0 & 3 & 3 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 0 & 1 & 1 & -\frac{2}{3} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 1 & 1 & -\frac{2}{3} \end{array}\right)$$

Consequently, $A^{-1} = \begin{pmatrix} 1 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix}$.

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$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & -2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & -1 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right)$$

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Consequently, $A^{-1} = \begin{pmatrix} 1 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix}$.

Try to find

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

## Definition

Let $A$ be a square matrix of type $(1,1)$ – i.e., $A = \begin{pmatrix} 1 \end{pmatrix}$ for some $a \in \mathbb{R}$. Then we say that the determinant of such matrix $A$ is $\det A = a$. Let $A = (a_{ij})_{i,j=1}^{n}$ be a square matrix of type $(n,n)$. We denote by $M_{ij}$ the determinant of the matrix $(n-1, n-1)$ which arises from $A$ by leaving out the $i-$th row and $j-$th column. Choose $k \in \{1, \ldots, n\}$. Then

$$\begin{aligned}\det A = (-1)^{k+1} a_{k1} M_{k1} + (-1)^{k+2} a_{k2} M_{k2} + \ldots + (-1)^{k+n} a_{kn} M_{kn} \\ = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}.\end{aligned}$$

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## Lemma

*It holds that* $\det A = \det A^T$.

## Examples

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then $\det A = a_{11}a_{22} - a_{12}a_{21}$.

**Examples**

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Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then

$$\begin{aligned} \det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \end{aligned}$$

## Examples

- Compute

$$\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

- Compute

$$\det \begin{pmatrix} 2 & 2 & 1 & -1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 3 & -2 & 1 \end{pmatrix}$$

## Determinant and elementary transformations

- Let $B$ arise from $A$ by multiplying one row by a real number $\alpha$. Then $\alpha \det A = \det B$.
- Let $B$ arise from $A$ by interchanging of two rows. Then $\det A = -\det B$.
- Let $B$ arise from $A$ by adding $\alpha$–multiple of one row to another one. Then $\det A = \det B$.

**Lemma**

*Let $A$ be a square matrix in the echelon form. Then the determinant of $A$ is a product of entries on the main diagonal, i.e.,* $\det A = a_{11}a_{22}\dots a_{nn}$.

**Reminder**

Compute

$$\det\begin{pmatrix}-2\end{pmatrix} = ?$$

$$\det\begin{pmatrix}2 & -3\\ 4 & 2\end{pmatrix} = ?$$

$$\det\begin{pmatrix}1 & 1 & 0\\ -1 & 0 & 1\\ 1 & 1 & 1\end{pmatrix} = ?$$

$$\det\begin{pmatrix}-1 & 1 & 0 & 2\\ 0 & 3 & -3 & 1\\ 2 & -3 & 0 & 2\\ 0 & 0 & 3 & -1\end{pmatrix} = ?$$

## Lemma

*Let $A$ be $n$ by $n$ matrix. The following statements are equivalent*

- *$A$ is singular.*
- $\det A = 0$.
- $Ax^T = 0$ *has a nontrivial solutions.*
- *The columns (or rows) of $A$ form a linearly dependent set.*
- *rank$A$ is strictly less than $n$.*

## Lemma

*Let $A$ be of type $(n, n)$. The following statements are equivalent*

- *$A$ is regular.*
- $\det A \neq 0$.
- $Ax^T = b^T$ *has unique solution for every right hand side $b$.*
- *The columns (or rows) of $A$ are linearly independent.*
- *rank$A = n$.*

Inverse matrix and determinant: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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be a regular matrix. Then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Solve

$$\begin{aligned} 2x - y &= 8 \\ -2x + 3y &= 12 \end{aligned}$$

**The Cramer rule**: Consider a system $Ax^T = b^T$. Assume $A$ is a regular $(n, n)$ matrix. Let $j = \{1, \dots n\}$ and denote $A_j$ the matrix arising from $A$ by replacing $j$–th column by a vector $b^T$. Then

$$x_j = \frac{\det A_j}{\det A}.$$

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$$x_j = \frac{\det A_j}{\det A}.$$

**Example**

Solve (by the Cramer rule)

$$\begin{aligned} 3x - 2y + 4z &= 3 \\ -2x + 5y + z &= 5 \\ x + y - 5z &= 0 \end{aligned}$$

**Eigenvectors and eigenvalues:**

Let $A$ be a square matrix. We are looking for $\lambda$ for which there is a nontrivial solution to

$$Ax^T = \lambda x^T.$$

Such number $\lambda$ is called **eigenvalue**.

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Such number $\lambda$ is called **eigenvalue**. This means that

$$(A - \lambda I)x^T = 0.$$

This equation has a nontrivial solution only if $A - \lambda I$ is a singular matrix. Consequently, $\lambda$ is an eigenvalue iff

$$\det(A - \lambda I) = 0.$$

The polynomial $\det(A - \lambda I)$ is called a **characteristic polynomial**.

Let $\lambda$ be an eigenvalue of $A$. A vector $v$ solving

$$(A - \lambda I)v = 0$$

is called eigenvector.

**Exercise**

Find all eigenvalues and eigenvectors to $A = \begin{pmatrix} 5 & 1 \\ 4 & 5 \end{pmatrix}$

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**Exercise**: Find eigenvalues and eigenvectors to $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

**Exercise**: Find all eigenvectors and eigenvalues to $A = \begin{pmatrix} -2 & -8 \\ 1 & 2 \end{pmatrix}$.

**Generalized eigenvectors**

A generalized eigenvector $w$ corresponding to an eigenvalue $\lambda$ is a vector satisfying

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## Generalized eigenvectors

A generalized eigenvector $w$ corresponding to an eigenvalue $\lambda$ is a vector satisfying

$$(A - \lambda I)w^T = v^T.$$

**Exercise**: Find all 2 eigenvectors (including the generalized one) of a matrix $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$.

**Exercise**: Find all 3 eigenvectors (including the generalized one) of a matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

Applications:

A population of rabbits has the following characteristics:

1. Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3.
2. During the first year, the rabbits produce no offspring. The average number of offspring per parent is 6 during the second year and 8 during the third year.

The population now consists of 24 rabbits in the first age, 24 rabbits in the second and 20 rabbits in the third. How many rabbits will there be in each age class in 1 year? Find a stable age distribution for the population of rabbits.

## Definition

Let $A$ be an $n$ by $n$ symmetric matrix. The mapping

$$Q: \begin{array}{c} \mathbb{R}^n \to \mathbb{R} \\ v \mapsto vAv^T \end{array}$$

is called a **quadratic form**.

## Definition

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is called a **quadratic form**.

**Examples**

- Quadratic form given by a matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is

$$(x, y) \mapsto \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 2xy + y^2$$

and we write $Q(x, y) = x^2 - 2xy + y^2$.

## Examples (sequel):

- A matrix $A$ associated with the quadratic form

$$Q(x, y, z) = x^2 - 3xz + y^2 - z^2$$

**Examples (sequel):**

- A matrix $A$ associated with the quadratic form

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is $A = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & -1 \end{pmatrix}$.

**Examples (sequel):**

- A matrix $A$ associated with the quadratic form

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- A quadratic form given by $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ is
$$Q(x,y,z) = x^2 - y^2 - 2z^2 + 4xz + 2yz.$$

## Definition

A quadratic form $Q$ is

- positive-definite if $Q(v) > 0$ for every $v \in \mathbb{R}^n \setminus \{0\}$
- positive-semidefinite if $Q(v) \geq 0$ for every $v \in \mathbb{R}^n$
- negative-definite if $Q(v) < 0$ for every $v \in \mathbb{R}^n \setminus \{0\}$
- negative-semidefinite if $Q(v) \leq 0$ for every $v \in \mathbb{R}^n$
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**Exercise:** Decide about the definiteness of the following quadratic forms:

- $Q(x, y) = x^2 - 2xy + y^2$
- $Q(x, y) = x^2 - y^2$
- $Q(x, y) = x^2 + 2xy + 2y^2$
- $Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

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**Sylvester rule**:
Let $A$ be $n$ by $n$ matrix. Denote $D_0 = 1$, $D_1 = \det(a_{11})$, $D_2 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ldots, D_n = \det A$ and assume $D_0, D_1, \ldots, D_n \neq 0$. If all products $D_0 \cdot D_1$, $D_1 \cdot D_2, \ldots, D_{n-1}D_n$ are positive, $A$ is a positive-definite matrix. If all the products are negative, $A$ is a negative-definite matrix. Otherwise it is indefinite matrix.

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**Exercise:**

- Verify that $Q(x, y) = x^2 + 2xy + 2y^2$ is positive-definite.
- Decide about the definiteness of $Q(x, y) = -x^2 - y^2$.

**Exercises** Decide about the definiteness of the following matrices

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

# Summary of linear Albegra

**Vector spaces:**

Vectors (sum and multiplication by scalar).

Linear combination of a set of vectors $\{v_i, i = 1 \dots n\}$ is any vector of a form

$$\sum_{i=1}^{n} \alpha_i v_i$$

where $\alpha_i \in \mathbb{R}$. All of such vectors form a linear span of $\{v_i, i = 1 \dots n\}$.

**Example**:

- Does $u = (1, -1, 0, 2)$ belong to the linear span of $v = (0, 1, 1, 0)$, $w = (-1, 1, 0, 1)$ and $x = (2, 2, 1, 1)$?

Vectors $\{v_i, i = 1 \ldots n\}$ are said to be linearly independent if the equation

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

has the only solution $\alpha_i = 0$ for every $\{i = 1 \ldots n\}$.

**Example**:

- Decide whether vectors $u = (2, 1, 1, 3)$, $v = (1, 1, 0, 2)$ and $w = (0, -1, 1, -1)$ are linearly dependent or independent.

Vectors $v_i,\ i = 1, \dots n$ generate a vector space $V$ if every vector $v \in V$ is a linear combination of $\{v_i,\ i = 1, \dots, n\}$.
An independent set of generators is called basis. The number of vectors in basis is called dimension.
Let $\{v_i,\ i = 1, \dots, n\}$ be a basis of $V$ and let $v \in V$. Then $v$ can be expressed as a linear combination of $v_i$ and the coefficients of the linear combination are called coordinates with respect to the given basis.

**Example**

- Form vectors $u = (1, 2, 1)$, $v = (1, 0, 1)$ and $w = (-1, 1, -2)$ a basis of $\mathbb{R}^3$? If yes, find the coordinates of $(2, 0, 0)$.

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**Example**

- What is the rank of

$$\begin{pmatrix} 3 & -2 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -2 & 2 & -1 \end{pmatrix}?$$

**System of equations** Solution by the Gauss elimination (transformation into the echelon form): Find all solutions to

$$
\begin{aligned}
y - x - t - z &= -3\\
2y - 2z - 2t &= 2x - 4\\
2x + y + 2z - t &= 0\\
x - 3y + z + 3t &= 7
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**Recall:** Pivots on position of the dependent variable.

**Square matrices:**

Identity matrix, inverse matrix, regular and singular matrix, determinants

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**Eigenavalues and eigenvectors:**

Nontrivial solutions to

$$(A - \lambda I)\, v = 0.$$

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**Example** Find eigenvalues and eigenvectors of

$$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

**Quadratic forms:**

Quadratic form is a function which maps $\mathbb{R}^n$ to $\mathbb{R}$ in such a way that every $v \in \mathbb{R}^n$ is mapped to $vAv^T$ for some symmetric matrix $A$.

**Examples**

- Write the quadratic form whose associated matrix is

$$\begin{pmatrix} 1 & -1 & \frac{1}{2} \\ -1 & -1 & 2 \\ \frac{1}{2} & 2 & -2 \end{pmatrix}$$

Write the matrix associated to

$$Q(x, y, z, t) = x^2 + 2y^2 - z^2 - t^2 + 4xy - 6yz + 3zt.$$

**Definitness:**

Let $A$ be a square symmetric matrix. Sign of a number $vAv^T$?

Sylvester rule.

**Example** Decide about the definiteness of

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

**That's all folks**