# Sequences and series

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Several examples: $a_n = \frac{1}{n}$, $a_n = n$, $a_n = 1$ and so on... Two ways how to set a sequence:

- Explicit formula – relation for each term, for example $a_n = 3(n-1) + 2$.
- Implicit formula – the first term (or first few terms) is given and there is a relation how to compute a $n-$th term from the previous one. For example $a_1 = 1$, $a_{n+1} = (n+1)a_n$ or $b_1 = b_2 = 1$ and $b_{n+2} = b_{n+1} + b_n$ (the famous Fibonacci sequence).

**Example**

- Find an explicit formula for $a_1 = 1$, $a_{n+1} = a_n + 2n + 1$ and verify your claim.

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Monotonicity – similarly to functions, anyway

## Definition

We say that $a_n$ is

- increasing, if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$,
- decreasing, if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$,
- non-decreasing, if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,
- non-increasing, if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

If $\{a_n\}_{n=1}^{\infty}$ posses one of these properties, we say it is monotone.

**Examples**

- $a_n = 1 - \frac{1}{n}$,
- $a_n = \frac{n^2}{2^n}$.

## Definition

Let $a_n$ be a sequence. A number $A \in \mathbb{R}$ is called a limit of $a_n$ if

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n > n_0,\ |a_n - A| < \varepsilon.$$

We write $\lim a_n = A$.

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A limit of $a_n$ is $+\infty$ if

$$\forall M > 0, \ \exists n_0 \in \mathbb{N}, \ \forall n > n_0, \ a_n > M.$$

In that case we write $\lim a_n = +\infty$.
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**Examples**

- $a_n = \frac{1}{n}$
- $a_n = n$
- $a_n = q^n,\ q > 1.$

## Observation

*Let $a_n$ be a sequence and let $A \in \mathbb{R}^*$ be its limit. Then it is determined uniqely.*

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## Proof.

Assume, for simplicity, that there are two real numbers $A$ and $B$ such that $A \neq B$ and $\lim a_n = A$ and $\lim a_n = B$. Take $\varepsilon = \frac{1}{3}|A - B|$. There exists $n_0 \in \mathbb{N}$ such that $|a_n - A| < \varepsilon$ and $|a_n - B| < \varepsilon$ for all $n > n_0$. Then, necessarily,

$$|A - B| < |A - a_n| + |B - b_n| < \frac{2}{3}\varepsilon < \frac{2}{3}|A - B|$$

which is a contradiction. □

## Lemma (Arithmetic of limits)

*Let $a_n$ and $b_n$ be sequences and let $c \in \mathbb{R}$. Then*

$$\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$$
$$\lim (a_n b_n) = \lim a_n \cdot \lim b_n$$
$$\lim c a_n = c \lim a_n$$
$$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$$

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*assuming the right hand side has meaning.*

Recall indefinite terms

$$\infty - \infty,\ \frac{\infty}{\infty},\ 0 \cdot \infty,\ \frac{0}{0},\ 1^{\infty},\ \infty^0,\ 0^0$$

which do not have any meaning. We also recall that $\frac{1}{\infty} = 0$.

## Proof.

We will assume (for simplicity) $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$. Moreover, we consider only $\lim(a_n + b_n) = \lim a_n + \lim b_n$. Take $\varepsilon > 0$ arbitrarily. Since $\lim a_n = A$ and $\lim b_n = B$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - A| < \frac{1}{2}\varepsilon$ and $|b_n - B| < \frac{1}{2}\varepsilon$. Consequently,

$$|a_n + b_n - A - B| \leq |a_n - A| + |b_n - B| < \varepsilon$$

and we have just verified that $A + B$ is a limit of $a_n + b_n$. □

**Examples**

- $\lim q^n$, $q \in (0,1)$
- $\lim n^2 - n$
- $\lim \frac{n+1}{n^2+3}$
- $\lim \frac{n^3+3n^2}{3n^3+n^2}$

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*Let $a_n$ be a sequence with real (finite) limit $A$. Then $a_n$ is a bounded sequence.*

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## Proof.

Indeed, take (for instance) $\varepsilon = 1$. There exists $n_0 \in \mathbb{N}$ such that $\{a_n\}_{n>n_0}$ is bounded from above by $A+1$ and from below by $A-1$. Next, $\{a_1, a_2, \ldots, a_{n_0}\}$ is a finite set and thus it is bounded from above (say by $M \in \mathbb{R}$) and from below by $m \in \mathbb{R}$. Then, $\{a_n\}_{n=1}^{\infty}$ is bounded from above by $\max\{M, A+1\}$ and from below by $\min\{m, A-1\}$. □

## Lemma (Sandwich lemma)

*Let* $a_n$, $b_n$, $c_n$ *be such that* $a_n \leq b_n \leq c_n$ *for all* $n \in \mathbb{N}$. *Assume, moreover, that* $\lim a_n = \lim c_n = A \in \mathbb{R}^*$. *Then* $\lim b_n$ *exists and* $\lim b_n = A$.

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*Let $a_n$, $b_n$, $c_n$ be such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Assume, moreover, that $\lim a_n = \lim c_n = A \in \mathbb{R}^*$. Then $\lim b_n$ exists and $\lim b_n = A$.*

## Proof.

Take an arbitrary $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $|a_n - A| < \varepsilon$ and $|c_n - A| < \varepsilon$. There may appear one of the following cases:

- $A \geq c_n$. In that case, $|b_n - A| < |a_n - A| < \varepsilon$.
- $A \leq a_n$. In that case, $|b_n - A| < |c_n - A| < \varepsilon$.
- $A \in (a_n, c_n)$. In that case, since $b_n \in [a_n, c_n]$, we have $|b_n - A| < |a_n - c_n| = |a_n - A + A - c_n| \leq |a_n - A| + |b_n - B| < 2\varepsilon$.

No matter which one is true, we have $|b_n - A| < 2\varepsilon$ and $A$ is a limit of $b_n$ according to the definition of limit. □

## Examples

- $\lim \frac{\cos n}{n}$
- $\lim \sqrt[n]{5^n + 3^n + 2}$ (Hint: $\lim \sqrt[n]{a} = 1$ for every $a > 0$.)

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Let $a_n$ be a sequence and let $k : \mathbb{N} \mapsto \mathbb{N}$ be an increasing sequence of natural numbers. Then $a_{k_n}$ is a *subsequence*.

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### Observation

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### Proof.

Once again, we assume for simplicity that $A \in \mathbb{R}$. For arbitrary $\varepsilon > 0$ there exists $n_0$ such that $|a_n - A| < \varepsilon$. However, as $k_n$ is an increasing sequence of natural numbers, there exists $n_1 \in \mathbb{N}$ such that $k_n > n_0$ whenever $n > n_1$. That means that for ever $n > n_1$ we have $|a_{k_n} - A| < \varepsilon$. The proof is complete. □

**Examples**

- $\lim(-1)^n(n+1)$
- $\lim q^n,\ q < 0.$
- $\lim \cos(\pi n)\frac{n-1}{n^2}$

## Example

- $\lim \frac{n^2}{2^n}$

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## Theorem (Heine)

*Let $c \in \mathbb{R}^*$ and let $d \in \mathbb{R}^*$. Then $\lim_{x\to c} f(x) = d$ if and only if $\lim f(x_n) = d$ for every sequence $x_n$ such that $\lim x_n = c$.*

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**Example**

- $\lim \frac{6n^2+(-1)^n}{(n+1)^3-n^3}$

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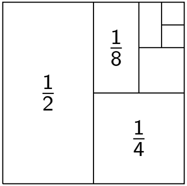

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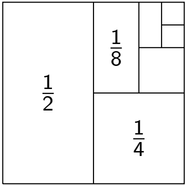


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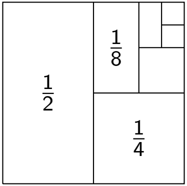


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On the other hand

$$1 + 1 + 1 + 1 + \ldots = \infty$$

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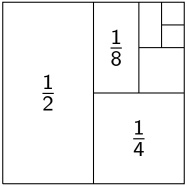


$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots<\infty$$

On the other hand

$$1+1+1+1+\ldots=\infty$$

### Definition

A **series** is a sum of infinitely many numbers.

Recall

$$\begin{aligned}(q+1)(q-1) &= q^2-1\\ (q^2+q+1)(q-1) &= q^3-1\\ \left(q^n+q^{n-1}+q^{n-2}+\ldots+q+1\right)(q-1) &= q^{n+1}-1.\end{aligned}$$

for every $q \in \mathbb{R}$.

Recall

$$(q+1)(q-1) = q^2 - 1$$
$$(q^2+q+1)(q-1) = q^3 - 1$$
$$\left(q^n + q^{n-1} + q^{n-2} + \ldots + q + 1\right)(q-1) = q^{n+1} - 1.$$

for every $q \in \mathbb{R}$.
Thus

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}.$$

for every $q \in (-1,1)$.

## Definition

Let $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ be a sequence. We define the *n−th partial sum*

$$s_n = \sum_{i=0}^{n} a_i.$$

If $\lim_{n\to\infty} s_n$ exists and is finite, than we say that $\sum_{i=0}^{\infty} a_i$ *converges* and its value is $\lim_{n\to\infty} s_n$. If a sum does not converge, we say that it *diverges*.

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**Examples**

- What is the second, third and fourth partial sum of $\sum_{i=0}^{\infty}(-1)^i$. In general, what is its $n-$th partial sum? Does the sum converge?
- What is the second, third and fourth partial sum of $\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+1}\right)$. In general, what is its $n-$th partial sum? Does the sum converge?

## Observation

*Let* $\sum_{i=0}^{\infty} a_i$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

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Proof: It holds that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = 0$$

where the last equality is true because of the arithmetic of limits.

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**Examples**

- $\sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \ldots$ diverge.
- How about $\sum_{n=1}^{\infty} \frac{1}{n}$?

**Series of positive numbers**

During this part, we suppose $a_n \geq 0$ for every $n \in \{0, 1, 2, 3 \ldots\}$.

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## Theorem

*Let* $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *and* $\{b_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *fulfill* $a_n \leq b_n$ *for every* $n \in \{0, 1, 2, 3, \ldots\}$*. Then*

- *if* $\sum_{n=0}^{\infty} b_n$ *converges, then also* $\sum_{n=0}^{\infty} a_n$ *converges,*
- *if* $\sum_{n=0}^{\infty} a_n$ *diverges, then also* $\sum_{n=0}^{\infty} b_n$ *diverges.*

**Example**

- $\sum_{n=1}^{\infty} \frac{2^n+n}{5^n}$
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Scales:

- It holds that
$$\sum_{n=0}^{\infty} q^n$$
converges for $q \in (0,1)$ and diverges for $q \geq 1$.
- It holds that
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
converges for $p > 1$ and diverges for $p \leq 1$.

## Limit version of the criterion

### Theorem

*Let* $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *and* $\{b_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *fulfill*

- $\lim \frac{a_n}{b_n} \in (0, \infty)$, *then* $\sum_{n=0}^{\infty} a_n$ *converge if and only if* $\sum_{n=0}^{\infty} b_n$ *converge,*
- $\lim \frac{a_n}{b_n} = 0$, *then if* $\sum_{n=0}^{\infty} b_n$ *converge then also* $\sum_{n=0}^{\infty} a_n$ *converge,*
- $\lim \frac{a_n}{b_n} = \infty$, *then if* $\sum_{n=0}^{\infty} a_n$ *converge, then also* $\sum_{n=0}^{\infty} b_n$ *converge.*

**Examples**

- Decide about the convergence of $\sum_{n=1}^{\infty} \frac{n^2+3n}{(n^3+1)^{3/2}}$.
- Decide about the convergence of $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$.

## Reminder
## Examples

- Let the sequence $\sum_{n=1}^{\infty} a_n$ has partial sums of the form

$$s_n = \frac{5 + 8n^2}{2 - 7n^2}.$$

Decide about the convergence of the series.

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## Examples

- Let the sequence $\sum_{n=1}^{\infty} a_n$ has partial sums of the form
$$s_n = \frac{5+8n^2}{2-7n^2}.$$
Decide about the convergence of the series.
- Decide about the convergence of the following series
$$\sum_{n=1}^{\infty} \frac{6+8n+9n^2}{3+2n+n^2}, \qquad \sum_{n=0}^{\infty} 3^{2+n}2^{1-3n}, \qquad \sum_{n=1}^{\infty} \frac{(-6)^n}{8^{2-n}},$$
$$\sum_{n=5}^{\infty} \frac{3ne^n}{n^2+1}, \qquad \sum_{n=4}^{\infty} \frac{10}{n^2-4n+3}, \qquad \sum_{n=1}^{\infty} \frac{n-1}{\sqrt{n^6+1}}.$$

## Observation (The d'Alambert criterion – ration test)

*Let* $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *be a sequence of positive real numbers. Then*

- *if* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ *then* $\sum_{n=0}^{\infty} a_n$ *converges,*
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**Example**

- Examine the convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}.$$

## Observation (The Cauchy criterion – root test)

*Let* $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *be a sequence of positive real numbers. Then*

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- Examine
$$\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}.$$

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- Rarely, one can tell the exact value of a series ($\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for $q \in (-1, 1)$)
- Rather than that, we focus on finiteness of a series – convergence vs divergence.
- Necessary condition for convergence: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim a_n = 0$.
- Are all summands non-negative (non-positive)?
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**Exercises:**

- Does $\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$ converge or diverge?
- Does $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$ converge or diverge?

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## Observation

*Let $\sum_{n=0}^{\infty} a_n$ converge absolutely. Then it converges.*

**Example**

- Examine $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

## Theorem (The Leibnitz criterion)

*Let* $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *be a sequence of positive numbers such that*

- $\lim_{n\to 0} a_n = 0.$
- $a_n$ *is a monotone sequence.*

*Then,*

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

*converges.*

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**Example**

- Examine $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$.
- Examine $\sum_{n=1}^{\infty} \frac{(-1)^n(1+(-1)^n)}{n}$.

**Gordon's growth model**: Shares bought at a time $t = 0$ for $P_0$ give us at time $t = 1$ the following return $r$

$$r = \frac{Div_1 + P_1 - P_0}{P_0}$$

where $Div_1$ is the dividend paid during the first year. We deduce

$$P_0 = \frac{Div_1}{1+r} + \frac{P_1}{1+r}.$$

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Consequently

$$P_0 = \sum_{t=1}^{\infty} \frac{Div_t}{(1+r)^t}.$$

We assume constant growth of the dividends, in particular we assume $Div_1$ given and $Div_t = (1+g) \cdot Div_{t-1}$.

**Gordon's growth model**: Shares bought at a time $t = 0$ for $P_0$ give us at time $t = 1$ the following return $r$

$$r = \frac{Div_1 + P_1 - P_0}{P_0}$$

where $Div_1$ is the dividend paid during the first year. We deduce

$$P_0 = \frac{Div_1}{1 + r} + \frac{P_1}{1 + r}.$$

Consequently

$$P_0 = \sum_{t=1}^{\infty} \frac{Div_t}{(1 + r)^t}.$$

We assume constant growth of the dividends, in particular we assume $Div_1$ given and $Div_t = (1 + g) \cdot Div_{t-1}$.Consequently

$$P_0 = \frac{Div_1}{(r - g)}$$

It holds that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$