Sequences and Series

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Sequences
Example

- A firm has purchased an item on a fixed payment plan of \$ 20 000 per year for 8 years. Payments are to be made at the beginning of each year. What is the present value of the total cash flow of payments for an interest rate of 20% per year?

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- What if the firm from the previous exercise has to pay \$ 20 000 yearly for ever?

Definition

A function $a : \mathbb{N} \to \mathbb{R}$, $\mathrm{Dom}\, a = \mathbb{N}$ is called a sequence. We write a_n instead of $a(n)$. The whole function is denoted by $\{a_n\}_{n=1}^{\infty}$.

Examples

- Write the first five elements of $a_n = \frac{1}{n}$. What is its fiftieth element?
- Write the first five elements of $a_n = 3(n-1)+2$. Write its twentieth element.
- Write the first five elements of $a_{n+1} = 2a_n + 1$, $a_1 = 1$. Is it possible to write directly its thirtieth element? Try to deduce its explicit formula and prove it is correct.
- Write the first six elements of $a_{n+2} = a_{n+1} + a_n$, $a_1 = a_0 = 1$.

Definition

We say that a_n is

- increasing, if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$,
- decreasing, if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$,
- non-decreasing, if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,
- non-increasing, if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

If $\{a_n\}_{n=1}^{\infty}$ posses one of these properties, we say it is monotone.

Exercises: Decide about the monotonicity of

- $a_n = \frac{n-1}{n}$
- $a_n = \frac{\sqrt{n}}{n+1}$
- $a_n = \sqrt{n+4} - \sqrt{n}$

Boundedness is similarl to functions, i.e.,

Definition

The sequence $\{a_n\}$ is bounded from above if there is $M \in \mathbb{R}$ such that $a_n \leq M$ for every $n \in \mathbb{N}$. Similarly, it is bounded from below if there is $m \in \mathbb{R}$ such that $a_n \geq m$ for every $n \in \mathbb{N}$.

Exercises:

Decide about the boundedness of the sequences from the previous slide, i.e.,

- $a_n = \frac{n-1}{n}$
- $a_n = \frac{\sqrt{n}}{n+1}$
- $a_n = \sqrt{n+4} - \sqrt{n}$

Limits, reminder

Definition

Let a_n be a sequence. A number $A \in \mathbb{R}$ is called a limit of a_n if

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n > n_0,\ |a_n - A| < \varepsilon.$$

We write $\lim a_n = A$.
A limit of a_n is $+\infty$ if

$$\forall M > 0,\ \exists n_0 \in \mathbb{N},\ \forall n > n_0,\ a_n > M.$$

In that case we write $\lim a_n = +\infty$.
A limit of a_n is $-\infty$ if $\lim -a_n = +\infty$. We then write $\lim a_n = -\infty$.

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Uniqueness, arithmetics of limits, and sandwich lemma work similarly to the limits of functions.

Exercises:

Compute

- $\lim \frac{n+1}{n^2+3}$
- $\lim \frac{n^3+3n^2}{3n^3+n^2}$

Definition

Let a_n be a sequence and let $k : \mathbb{N} \mapsto \mathbb{N}$ be an increasing sequence of natural numbers. Then a_{k_n} is a *subsequence.*

Lemma

Let a_n be a sequence such that $\lim a_n = A$, $A \in \mathbb{R}^*$. Then every subsequence a_{k_n} has a limit A.

Exercises:

Compute

- $\lim \frac{(-1)^n n^2+2}{n^2}$
- $\lim \frac{1}{n} \sin n$
- $\lim \sqrt[n]{5^n + 3^n + 2^n}$

On relation with limits of function

Theorem (Heine)

Let x_0 be a limit point of $\mathrm{Dom} f$. Then $\lim_{x\to x_0} f(x) = A$ if and only if it holds that $\lim f(x_n) = A$ for every x_n such that $\lim x_n = x$

Exercises:

Compute

- $\lim \left(1+\frac{1}{2n}\right)^{n-4}$.
- $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$.
- $\lim \frac{n^2}{a^n}$, $a > 1$.

Series

Back to the initial example:

- A firm has purchased an item on a fixed payment plan of \$ 20 000 per year for 8 years. Payments are to be made at the beginning of each year. What is the present value of the total cash flow of payments for an interest rate of 20% per year?
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- A firm has purchased an item on a fixed payment plan of $ 20 000 per year for 8 years. Payments are to be made at the beginning of each year. What is the present value of the total cash flow of payments for an interest rate of 20% per year?
- What if the firm from the previous exercise has to pay $ 20 000 yearly for ever?
- Imagine you face the decision whether you by your own apartment or you rent it. Either, you may pay 8 000 000 CZK for a one-bedroom apartment or you can rent it for 20 000 CZK per month (lets simplify it to 240 000 CZK per year) paid for ever. Assuming the interest rate is 4% per year which of these choices has less present value? What if the interest rate drops to 2%?

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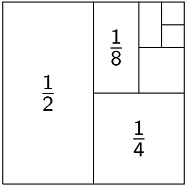

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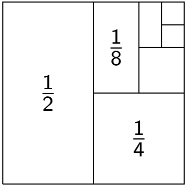

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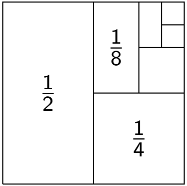

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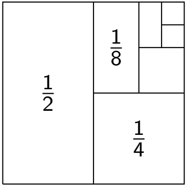

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On the other hand

$$1+1+1+1+\ldots=\infty$$

Definition

A **series** is a sum of infinitely many numbers.

Recall

$$(q+1)(q-1) = q^2 - 1$$
$$(q^2+q+1)(q-1) = q^3 - 1$$
$$\left(q^n + q^{n-1} + q^{n-2} + \ldots + q + 1\right)(q-1) = q^{n+1} - 1.$$

for every $q \in \mathbb{R}$.

Recall

$$\begin{aligned}(q+1)(q-1) &= q^2-1\\ (q^2+q+1)(q-1) &= q^3-1\\ \left(q^n+q^{n-1}+q^{n-2}+\ldots+q+1\right)(q-1) &= q^{n+1}-1.\end{aligned}$$

for every $q \in \mathbb{R}$.
Thus

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}.$$

for every $q \in (-1,1)$.

Definition

Let $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ be a sequence. We define the *n−th partial sum*

$$s_n = \sum_{i=0}^{n} a_i.$$

If $\lim_{n\to\infty} s_n$ exists and is finite, than we say that $\sum_{i=0}^{\infty} a_i$ *converges* and its value is $\lim_{n\to\infty} s_n$. If a sum does not converge, we say that it *diverges*.

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Examples

- What is the second, third and fourth partial sum of $\sum_{i=0}^{\infty}(-1)^i$. In general, what is its $n-$th partial sum? Does the sum converge?
- What is the second, third and fourth partial sum of $\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+1}\right)$. In general, what is its $n-$th partial sum? Does the sum converge?

Observation

Let $\sum_{i=0}^{\infty} a_i$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

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Examples

- $\sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \dots$ diverge.
- How about $\sum_{n=1}^{\infty} \frac{1}{n}$?

Series of positive numbers

During this part, we suppose $a_n \geq 0$ for every $n \in \{0, 1, 2, 3 \ldots\}$.

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Theorem

Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *and* $\{b_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *fulfill* $a_n \leq b_n$ *for every* $n \in \{0, 1, 2, 3, \dots\}$*. Then*

- *if* $\sum_{n=0}^{\infty} b_n$ *converges, then also* $\sum_{n=0}^{\infty} a_n$ *converges,*
- *if* $\sum_{n=0}^{\infty} a_n$ *diverges, then also* $\sum_{n=0}^{\infty} b_n$ *diverges.*

Example

- $\sum_{n=1}^{\infty} \frac{2^n+n}{5^n}$
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Scales:

- It holds that
$$\sum_{n=0}^{\infty} q^n$$
converges for $q \in (0,1)$ and diverges for $q \geq 1$.
- It holds that
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
converges for $p > 1$ and diverges for $p \leq 1$.

Limit version of the criterion

Theorem

Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *and* $\{b_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *fulfill*

- $\lim \frac{a_n}{b_n} \in (0, \infty)$, *then* $\sum_{n=0}^{\infty} a_n$ *converge if and only if* $\sum_{n=0}^{\infty} b_n$ *converge,*
- $\lim \frac{a_n}{b_n} = 0$, *then if* $\sum_{n=0}^{\infty} b_n$ *converge then also* $\sum_{n=0}^{\infty} a_n$ *converge,*
- $\lim \frac{a_n}{b_n} = \infty$, *then if* $\sum_{n=0}^{\infty} a_n$ *converge, then also* $\sum_{n=0}^{\infty} b_n$ *converge.*

Examples

- Decide about the convergence of $\sum_{n=1}^{\infty} \frac{n^2+3n}{(n^3+1)^{3/2}}$.
- Decide about the convergence of $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$.

Reminder

Examples

- Let the sequence $\sum_{n=1}^{\infty} a_n$ has partial sums of the form

$$s_n = \frac{5+8n^2}{2-7n^2}.$$

Decide about the convergence of the series.

Reminder

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  Decide about the convergence of the series.

- Decide about the convergence of the following series

$$\sum_{n=1}^{\infty} \frac{6+8n+9n^2}{3+2n+n^2}, \qquad \sum_{n=0}^{\infty} 3^{2+n}2^{1-3n}, \qquad \sum_{n=1}^{\infty} \frac{(-6)^n}{8^{2-n}},$$

$$\sum_{n=5}^{\infty} \frac{3ne^n}{n^2+1}, \qquad \sum_{n=4}^{\infty} \frac{10}{n^2-4n+3}, \qquad \sum_{n=1}^{\infty} \frac{n-1}{\sqrt{n^6+1}}.$$

Observation (The d'Alambert criterion – ration test)

Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *be a sequence of positive real numbers. Then*

- *if* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ *then* $\sum_{n=0}^{\infty} a_n$ *converges,*
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Example

- Examine the convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}.$$

Observation (The Cauchy criterion – root test)

Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *be a sequence of positive real numbers. Then*

- *if* $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ *then* $\sum_{n=0}^{\infty} a_n$ *converges,*
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- Examine

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}.$$

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- Rather than that, we focus on finiteness of a series – convergence vs divergence.
- Necessary condition for convergence: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim a_n = 0$.
- Are all summands non-negative (non-positive)?
  - Yes, then we may use: comparison, the ration test, the root test.
  - No, then we will see today.

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Exercises:

- Does $\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$ converge or diverge?
- Does $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$ converge or diverge?
- Does $\sum_{n=1}^{\infty} \frac{5^n}{3^{n+1}4^{2n-1}} - \frac{\sqrt{n}-\sqrt{n-1}}{n}$ converge or diverge?

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converges. Then we say that $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* (or converges absolutely)

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Let $\sum_{n=0}^{\infty} a_n$ converge absolutely. Then it converges.

Example

- Examine $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

Theorem (The Leibnitz criterion)

Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ *be a sequence of positive numbers such that*

- $\lim_{n\to 0} a_n = 0$.
- a_n *is a monotone sequence.*

Then,

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

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Example

- Examine $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$.
- Examine $\sum_{n=1}^{\infty} \frac{(-1)^n(1+(-1)^n)}{n}$.

Gordon's growth model: Shares bought at a time $t = 0$ for P_0 give us at time $t = 1$ the following return r

$$r = \frac{Div_1 + P_1 - P_0}{P_0}$$

where Div_1 is the dividend paid during the first year. We deduce

$$P_0 = \frac{Div_1}{1 + r} + \frac{P_1}{1 + r}.$$

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It holds that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$